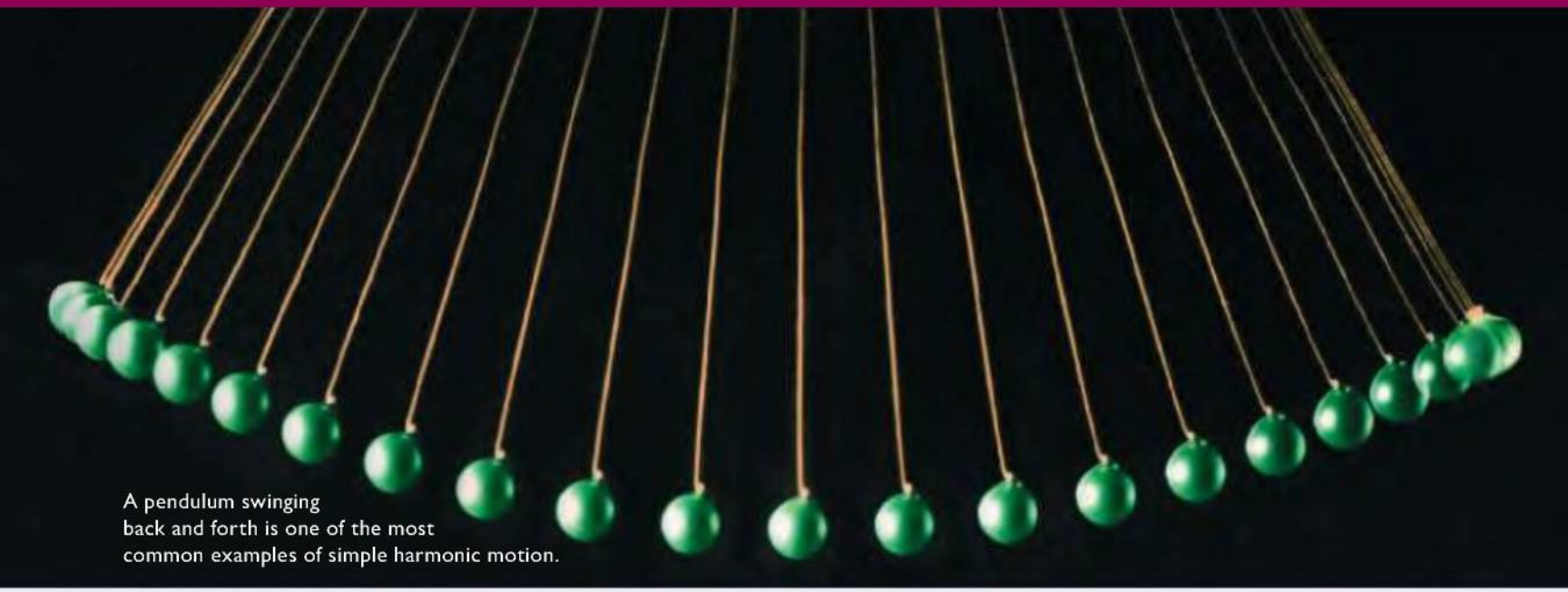
15 Oscillations

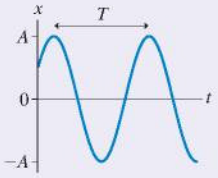


A pendulum swinging back and forth is one of the most common examples of simple harmonic motion.

IN THIS CHAPTER, you will learn about systems that oscillate in simple harmonic motion.

What are oscillations?

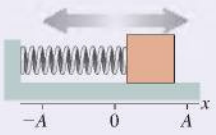
Oscillatory motion is a repetitive motion back and forth around an equilibrium position. We'll describe oscillations in terms of their amplitude, period, and frequency. The most important oscillation is simple harmonic motion (SHM), where the position and velocity graphs are sinusoidal.



What things undergo SHM?

The prototype of SHM is a mass oscillating on a spring. Lessons learned from this system apply to all SHM.

- A pendulum is a classic example of SHM.
- Any system with a linear restoring force undergoes SHM.

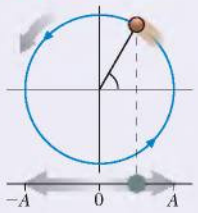


« LOOKING BACK Section 9.4 Restoring forces

How is SHM related to circular motion?

The projection of uniform circular motion onto a line oscillates back and forth in SHM.

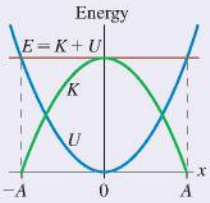
- This link to circular motion will help us develop the mathematics of SHM.
- A phase constant, based on the angle on a circle, will describe the initial conditions.



« LOOKING BACK Section 4.4 Circular motion

Is energy conserved in SHM?

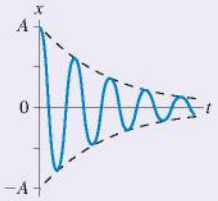
If there is no friction or other dissipative force, the mechanical energy of an oscillating system is conserved. Energy is transformed back and forth between kinetic and potential energy. Energy conservation is an important problem-solving strategy.



« LOOKING BACK Sections 10.3–10.5 Elastic potential energy and energy diagrams

What if there's friction?

If there's dissipation, the system "runs down." This is called a damped oscillation. The oscillation amplitude undergoes exponential decay. But the amplitude can grow very large when an oscillatory system is *driven* at its natural frequency. This is called resonance.



Why is SHM important?

Simple harmonic motion is one of the most common and important motions in science and engineering.

- Oscillations and vibrations occur in mechanical, electrical, chemical, and atomic systems. Understanding how a system might oscillate is an important part of engineering design.
- More complex oscillations can be understood in terms of SHM.
- Oscillations are the sources of waves, which we'll study in the next two chapters.

15.1 Simple Harmonic Motion

One of the most common and important types of motion is **oscillatory motion**—a repetitive motion back and forth around an equilibrium position. Swinging chandeliers, vibrating guitar strings, the electrons in cell phone circuits, and even atoms in solids are all undergoing oscillatory motion. In addition, oscillations are the sources of *waves,* our subject in the following two chapters. Oscillations play a major role in all fields of science and engineering.

FIGURE 15.1 shows position-versus-time graphs for two different oscillating systems. The shape of the graph depends on the details of the oscillator, but all oscillators have two things in common:

1. The oscillations take place around an equilibrium position.
2. The motion is *periodic,* repeating at regular intervals of time.

The time to complete one full cycle, or one oscillation, is called the **period** of the oscillation. Period is represented by the symbol T.

A system can oscillate in many ways, but the most fundamental oscillation is the smooth *sinusoidal* oscillation (i.e., like a sine or cosine) of Figure 15.1. This sinusoidal oscillation is called **simple harmonic motion,** abbreviated SHM. You'll learn in more advanced courses that *any* oscillation can be represented as a sum of sinusoidal oscillations, so SHM is the basis for understanding all oscillatory motion.

The prototype of simple harmonic motion is a mass oscillating on a spring. FIGURE 15.2 shows an air-track glider attached to a spring. If the glider is pulled out a few centimeters and released, it oscillates back and forth. The graph shows an actual air-track measurement in which the glider's position was recorded 20 times per second. This is a position-versus-time graph that has been rotated 90° from its usual orientation to match the motion of the glider. You can see that it's a sinusoidal oscillation—simple harmonic motion.

As Figures 15.1 and 15.2 show, an oscillator moves back and forth between $x = -A$ and $x = +A$, where A, the **amplitude** of the motion, is the maximum displacement from equilibrium. Notice that the amplitude is the distance from the *axis* to a maximum or minimum, *not* the distance from the minimum to the maximum.

Period and amplitude are two important characteristics of oscillatory motion. A third is the **frequency,** f, which is the number of cycles or oscillations completed per second. If one cycle takes T seconds, the oscillator can complete $1/T$ cycles each second. That is, period and frequency are inverses of each other:

$$f = \frac{1}{T} \quad \text{or} \quad T = \frac{1}{f} \tag{15.1}$$

The units of frequency are **hertz,** abbreviated Hz, named in honor of the German physicist Heinrich Hertz, who produced the first artificially generated radio waves in 1887. By definition,

$$1 \text{ Hz} \equiv 1 \text{ cycle per second} = 1 \text{ s}^{-1}$$

We will often deal with very rapid oscillations and make use of the units shown in TABLE 15.1. For example, electrons oscillating back and forth at 101 MHz in an FM radio circuit have an oscillation period $T = 1/(101 \times 10^6 \text{ Hz}) = 9.9 \times 10^{-9} \text{ s} = 9.9 \text{ ns}$.

NOTE Uppercase and lowercase letters *are* important. 1 MHz is 1 megahertz = 10^6 Hz, but 1 mHz is 1 millihertz = 10^{-3} Hz!

Kinematics of Simple Harmonic Motion

We'll start by *describing* simple harmonic motion mathematically—that is, with kinematics. Then in Section 15.4 we'll take up the dynamics of how forces *cause* simple harmonic motion.

FIGURE 15.1 Examples of oscillatory motion.

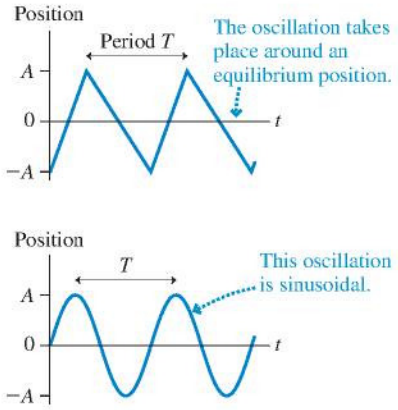


FIGURE 15.2 A prototype simple-harmonic-motion experiment.

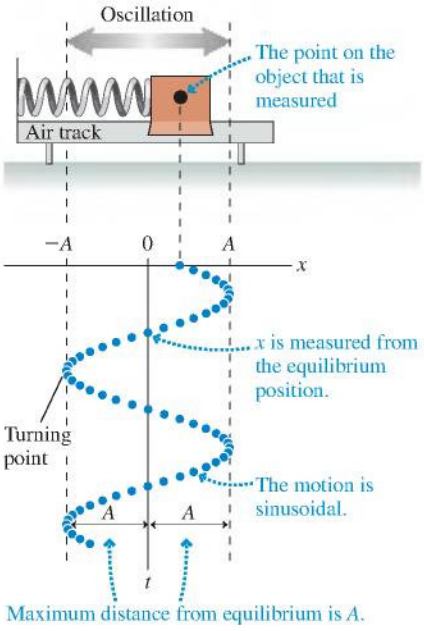


TABLE 15.1 Units of frequency

Frequency	Period
10^3 Hz = 1 kilohertz = 1 kHz	1 ms
10^6 Hz = 1 megahertz = 1 MHz	1 μs
10^9 Hz = 1 gigahertz = 1 GHz	1 ns

FIGURE 15.3 Position and velocity graphs for simple harmonic motion.

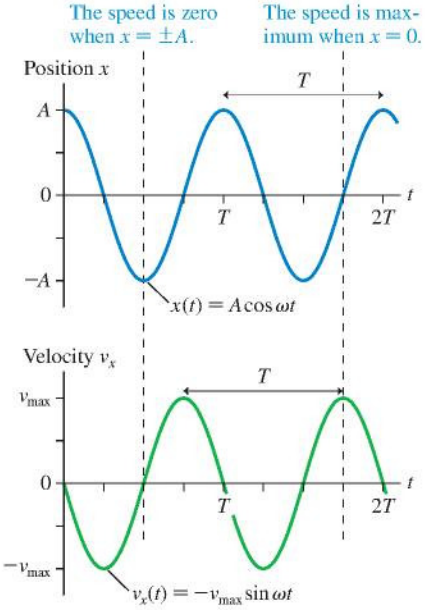


FIGURE 15.3 shows a SHM position graph—such as the one generated by the air-track glider—in its "normal" position. For the moment we'll assume that the oscillator starts at maximum displacement ($x = +A$) at $t = 0$. Also shown is the oscillator's velocity-versus-time graph, which we can deduce from the slope of the position graph.

- The instantaneous velocity is zero at the instants when $x = \pm A$ because the slope of the position graph is zero. These are the *turning points* in the motion.
- The position graph has maximum slope when $x = 0$, so these are points of maximum speed. When $x = 0$ with a positive slope—maximum speed to the right—the instantaneous velocity is $v_x = +v_{max}$, where v_{max} is the amplitude of the velocity curve. Similarly, $v_x = -v_{max}$ when $x = 0$ with a negative slope—maximum speed to the left.

Although these are empirical observations (we don't yet have any "theory" of oscillation), we can see that the position graph, with a maximum at $t = 0$, is a cosine function with amplitude A and period T. We can write this as

$$x(t) = A\cos\left(\frac{2\pi t}{T}\right) \tag{15.2}$$

where the notation $x(t)$ indicates that x is a *function* of time t.

NOTE The arguments of sine and cosine functions are in *radians*. This will be true throughout our study of oscillations and waves. Be sure to set your calculator to radian mode before doing calculations.

Because $\cos(0\text{ rad}) = \cos(2\pi\text{ rad}) = 1$, we can see that $x = A$ at $t = 0$ and again at $t = T$. In other words, this is a cosine function with amplitude A and period T. Notice that x passes through zero at $t = \frac{1}{4}T$ and $t = \frac{3}{4}T$ because $\cos\left(\frac{1}{2}\pi\right) = \cos\left(\frac{3}{2}\pi\right) = 0$.

We can write Equation 15.2 in two alternative forms. First, because the oscillation frequency is $f = 1/T$, we can write

$$x(t) = A\cos(2\pi ft) \tag{15.3}$$

Second, recall from Chapter 4 that a particle in circular motion has *angular velocity* ω that is related to the period by $\omega = 2\pi/T$, where ω is in rad/s. For oscillations and waves, ω is called the **angular frequency.** We can write the position in terms of ω as

$$x(t) = A\cos(\omega t) \tag{15.4}$$

Most of our work with oscillations and waves will be in terms of the angular frequency.

Now that we've defined $f = 1/T$, we see that ω, f, and T are related by

$$\omega\text{ (in rad/s)} = \frac{2\pi}{T} = 2\pi f\text{ (in Hz)} \tag{15.5}$$

Be careful! Both f and ω are frequencies, but they're not the same and they're not interchangeable. Frequency f is the true frequency, in cycles per second, and it's always measured in Hz. Angular frequency ω is useful because it's related to the angle of the cosine function and, as you'll learn in the next section, to a circular-motion analog of SHM, but it's always in rad/s.

Just as the position graph is a cosine function, you can see that the velocity graph in Figure 15.3 is an "upside-down" sine function with the same period. We can write the velocity function as

$$v_x(t) = -v_{max}\sin\left(\frac{2\pi t}{T}\right) \tag{15.6}$$

where the minus sign inverts the graph. This function is zero at $t = 0$ and again at $t = T$.

NOTE v_{max} is the maximum *speed* and thus is a *positive* number.

We deduced Equation 15.6 from the experimental results, but we could equally well find it from the position function of Equation 15.2. After all, velocity is the time derivative of position. TABLE 15.2 reminds you of the derivatives of the sine and cosine functions. Using the derivative of the position function, we find

$$v_x(t) = \frac{dx}{dt} = -\frac{2\pi A}{T}\sin\left(\frac{2\pi t}{T}\right) = -2\pi fA\sin(2\pi ft) = -\omega A\sin\omega t \qquad (15.7)$$

Comparing Equation 15.7, the mathematical definition of velocity, to Equation 15.6, the empirical description, we see that the maximum speed of an oscillation is

$$v_{max} = \frac{2\pi A}{T} = 2\pi fA = \omega A \qquad (15.8)$$

Not surprisingly, the object has a greater maximum speed if you stretch the spring farther and give the oscillation a larger amplitude.

TABLE 15.2 Derivatives of sine and cosine functions

$\frac{d}{dt}\big(a\sin(bt+c)\big) = +ab\cos(bt+c)$
$\frac{d}{dt}\big(a\cos(bt+c)\big) = -ab\sin(bt+c)$

EXAMPLE 15.1 A system in simple harmonic motion

An air-track glider is attached to a spring, pulled 20.0 cm to the right, and released at $t = 0$ s. It makes 15 oscillations in 10.0 s.

a. What is the period of oscillation?
b. What is the object's maximum speed?
c. What are the position and velocity at $t = 0.800$ s?

MODEL An object oscillating on a spring is in SHM.

SOLVE a. The oscillation frequency is

$$f = \frac{15\text{ oscillations}}{10.0\text{ s}} = 1.50\text{ oscillations/s} = 1.50\text{ Hz}$$

Thus the period is $T = 1/f = 0.667$ s.

b. The oscillation amplitude is $A = 0.200$ m. Thus

$$v_{max} = \frac{2\pi A}{T} = \frac{2\pi(0.200\text{ m})}{0.667\text{ s}} = 1.88\text{ m/s}$$

c. The object starts at $x = +A$ at $t = 0$ s. This is exactly the oscillation described by Equations 15.2 and 15.6. The position at $t = 0.800$ s is

$$x = A\cos\left(\frac{2\pi t}{T}\right) = (0.200\text{ m})\cos\left(\frac{2\pi(0.800\text{ s})}{0.667\text{ s}}\right)$$

$$= (0.200\text{ m})\cos(7.54\text{ rad}) = 0.0625\text{ m} = 6.25\text{ cm}$$

The velocity at this instant of time is

$$v_x = -v_{max}\sin\left(\frac{2\pi t}{T}\right) = -(1.88\text{ m/s})\sin\left(\frac{2\pi(0.800\text{ s})}{0.667\text{ s}}\right)$$

$$= -(1.88\text{ m/s})\sin(7.54\text{ rad}) = -1.79\text{ m/s} = -179\text{ cm/s}$$

At $t = 0.800$ s, which is slightly more than one period, the object is 6.25 cm to the right of equilibrium and moving to the *left* at 179 cm/s. Notice the use of radians in the calculations.

EXAMPLE 15.2 Finding the time

A mass oscillating in simple harmonic motion starts at $x = A$ and has period T. At what time, as a fraction of T, does the object first pass through $x = \frac{1}{2}A$?

SOLVE Figure 15.3 showed that the object passes through the equilibrium position $x = 0$ at $t = \frac{1}{4}T$. This is one-quarter of the total distance in one-quarter of a period. You might expect it to take $\frac{1}{8}T$ to reach $\frac{1}{2}A$, but this is not the case because the SHM graph is not linear between $x = A$ and $x = 0$. We need to use $x(t) = A\cos(2\pi t/T)$. First, we write the equation with $x = \frac{1}{2}A$:

$$x = \frac{A}{2} = A\cos\left(\frac{2\pi t}{T}\right)$$

Then we solve for the time at which this position is reached:

$$t = \frac{T}{2\pi}\cos^{-1}\left(\frac{1}{2}\right) = \frac{T}{2\pi}\frac{\pi}{3} = \frac{1}{6}T$$

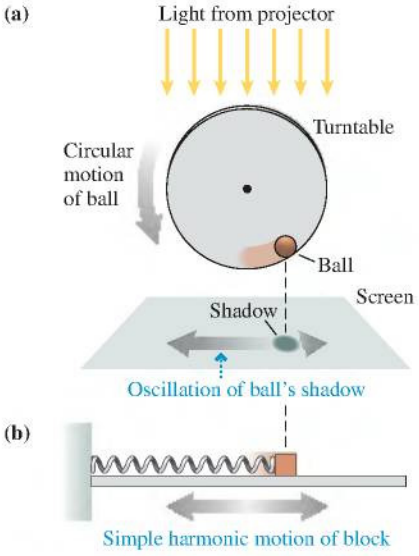
ASSESS The motion is slow at the beginning and then speeds up, so it takes longer to move from $x = A$ to $x = \frac{1}{2}A$ than it does to move from $x = \frac{1}{2}A$ to $x = 0$. Notice that the answer is independent of the amplitude A.

STOP TO THINK 15.1 An object moves with simple harmonic motion. If the amplitude and the period are both doubled, the object's maximum speed is

a. Quadrupled. b. Doubled. c. Unchanged.
d. Halved. e. Quartered.

15.2 SHM and Circular Motion

FIGURE 15.4 A projection of the circular motion of a rotating ball matches the simple harmonic motion of an object on a spring.



The graphs of Figure 15.3 and the position function $x(t) = A\cos\omega t$ are for an oscillation in which the object just happened to be at $x_0 = A$ at $t = 0$. But you will recall that $t = 0$ is an arbitrary choice, the instant of time when you or someone else starts a stopwatch. What if you had started the stopwatch when the object was at $x_0 = -A$, or when the object was somewhere in the middle of an oscillation? In other words, what if the oscillator had different *initial conditions?* The position graph would still show an oscillation, but neither Figure 15.3 nor $x(t) = A\cos\omega t$ would describe the motion correctly.

To learn how to describe the oscillation for other initial conditions it will help to turn to a topic you studied in Chapter 4—circular motion. There's a very close connection between simple harmonic motion and circular motion.

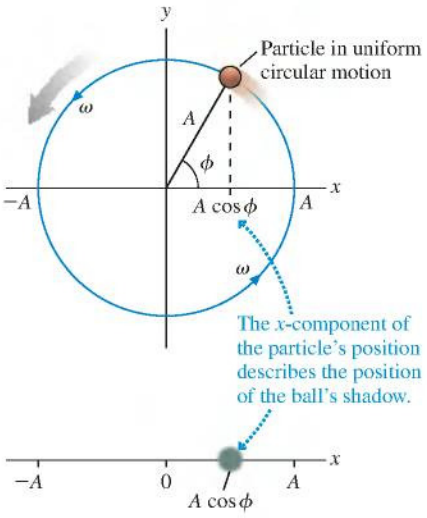
Imagine you have a turntable with a small ball glued to the edge. **FIGURE 15.4a** shows how to make a "shadow movie" of the ball by projecting a light past the ball and onto a screen. The ball's shadow oscillates back and forth as the turntable rotates. This is certainly periodic motion, with the same period as the turntable, but is it simple harmonic motion?

To find out, you could place a real object on a real spring directly below the shadow, as shown in **FIGURE 15.4b**. If you did so, and if you adjusted the turntable to have the same period as the spring, you would find that the shadow's motion exactly matches the simple harmonic motion of the object on the spring. **Uniform circular motion projected onto one dimension is simple harmonic motion.**

To understand this, consider the particle in **FIGURE 15.5**. It is in uniform circular motion, moving *counterclockwise* in a circle with radius A. As in Chapter 4, we can locate the particle by the angle ϕ measured counterclockwise (ccw) from the x-axis. Projecting the ball's shadow onto a screen in Figure 15.4 is equivalent to observing just the x-component of the particle's motion. Figure 15.5 shows that the x-component, when the particle is at angle ϕ, is

$$x = A\cos\phi \tag{15.9}$$

FIGURE 15.5 A particle in uniform circular motion with radius A and angular velocity ω.



Recall that the particle's *angular velocity,* in rad/s, is

$$\omega = \frac{d\phi}{dt} \tag{15.10}$$

This is the rate at which the angle ϕ is increasing. If the particle starts from $\phi_0 = 0$ at $t = 0$, its angle at a later time t is simply

$$\phi = \omega t \tag{15.11}$$

As ϕ increases, the particle's x-component is

$$x(t) = A\cos\omega t \tag{15.12}$$

This is identical to Equation 15.4 for the position of a mass on a spring! Thus the x-component of a particle in uniform circular motion is simple harmonic motion.

NOTE When used to describe oscillatory motion, ω is called the *angular frequency* rather than the angular velocity. The angular frequency of an oscillator has the same numerical value, in rad/s, as the angular velocity of the corresponding particle in circular motion.

The names and units can be a bit confusing until you get used to them. It may help to notice that *cycle* and *oscillation* are not true units. Unlike the "standard meter" or the "standard kilogram," to which you could compare a length or a mass, there is no "standard cycle" to which you can compare an oscillation. Cycles and oscillations are simply counted events. Thus the frequency f has units of hertz, where $1\ \text{Hz} = 1\ \text{s}^{-1}$. We may *say* "cycles per second" just to be clear, but the actual units are only "per second."

The radian is the SI unit of angle. However, the radian is a *defined* unit. Further, its definition as a ratio of two lengths ($\theta = s/r$) makes it a pure number without dimensions. As we noted in Chapter 4, the unit of angle, be it radians or degrees, is really just a *name* to remind us that we're dealing with an angle. The 2π in the equation $\omega = 2\pi f$ (and in similar situations), which is stated without units, *means* 2π rad/cycle. When multiplied by the frequency f in cycles/s, it gives the frequency in rad/s. That is why, in this context, ω is called the angular *frequency*.

NOTE *Hertz* is specifically "cycles per second" or "oscillations per second." It is used for f but *not* for ω. We'll always be careful to use rad/s for ω, but you should be aware that many books give the units of ω as simply s^{-1}.

Initial Conditions: The Phase Constant

Now we're ready to consider the issue of other initial conditions. The particle in Figure 15.5 started at $\phi_0 = 0$. This was equivalent to an oscillator starting at the far right edge, $x_0 = A$. FIGURE 15.6 shows a more general situation in which the initial angle ϕ_0 can have any value. The angle at a later time t is then

$$\phi = \omega t + \phi_0 \tag{15.13}$$

In this case, the particle's projection onto the x-axis at time t is

$$x(t) = A\cos(\omega t + \phi_0) \tag{15.14}$$

If Equation 15.14 describes the particle's projection, then it must also be the position of an oscillator in simple harmonic motion. The oscillator's velocity v_x is found by taking the derivative dx/dt. The resulting equations,

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_0) \\ v_x(t) &= -\omega A\sin(\omega t + \phi_0) = -v_{max}\sin(\omega t + \phi_0) \end{aligned} \tag{15.15}$$

are the two primary kinematic equations of simple harmonic motion.

The quantity $\phi = \omega t + \phi_0$, which steadily increases with time, is called the **phase** of the oscillation. The phase is simply the *angle* of the circular-motion particle whose shadow matches the oscillator. The constant ϕ_0 is called the **phase constant.** It is determined by the *initial conditions* of the oscillator.

To see what the phase constant means, set $t = 0$ in Equations 15.15:

$$\begin{aligned} x_0 &= A\cos\phi_0 \\ v_{0x} &= -\omega A\sin\phi_0 \end{aligned} \tag{15.16}$$

The position x_0 and velocity v_{0x} at $t = 0$ are the initial conditions. **Different values of the phase constant correspond to different starting points on the circle and thus to different initial conditions.**

The cosine function of Figure 15.3 and the equation $x(t) = A\cos\omega t$ are for an oscillation with $\phi_0 = 0$ rad. You can see from Equations 15.16 that $\phi_0 = 0$ rad implies $x_0 = A$ and $v_0 = 0$. That is, the particle starts from rest at the point of maximum displacement.

FIGURE 15.7, on the next page, illustrates these ideas by looking at three values of the phase constant: $\phi_0 = \pi/3$ rad (60°), $-\pi/3$ rad (−60°), and π rad (180°). Notice that $\phi_0 = \pi/3$ rad and $\phi_0 = -\pi/3$ rad have the same starting position, $x_0 = \frac{1}{2}A$. This is a property of the cosine function in Equation 15.16. But these are *not* the same initial conditions. In one case the oscillator starts at $\frac{1}{2}A$ while moving to the left, in the other case it starts at $\frac{1}{2}A$ while moving to the right. You can distinguish between the two by visualizing the motion.

All values of the phase constant ϕ_0 between 0 and π rad correspond to a particle in the upper half of the circle and *moving to the left.* Thus v_{0x} is negative. All values of the phase constant ϕ_0 between π and 2π rad (or, as they are usually stated, between $-\pi$ and 0 rad) have the particle in the lower half of the circle and *moving to the right.* Thus v_{0x} is positive. If you're told that the oscillator is at $x = \frac{1}{2}A$ and moving to the right at $t = 0$, then the phase constant must be $\phi_0 = -\pi/3$ rad, not $+\pi/3$ rad.

FIGURE 15.6 A particle in uniform circular motion with initial angle ϕ_0.

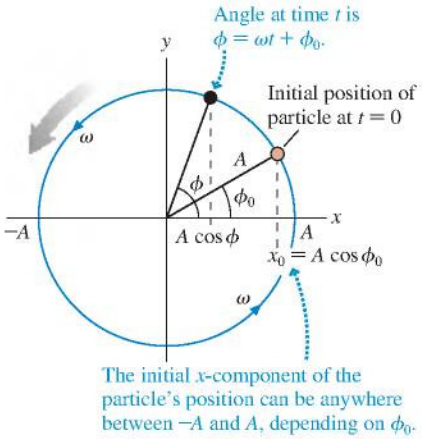
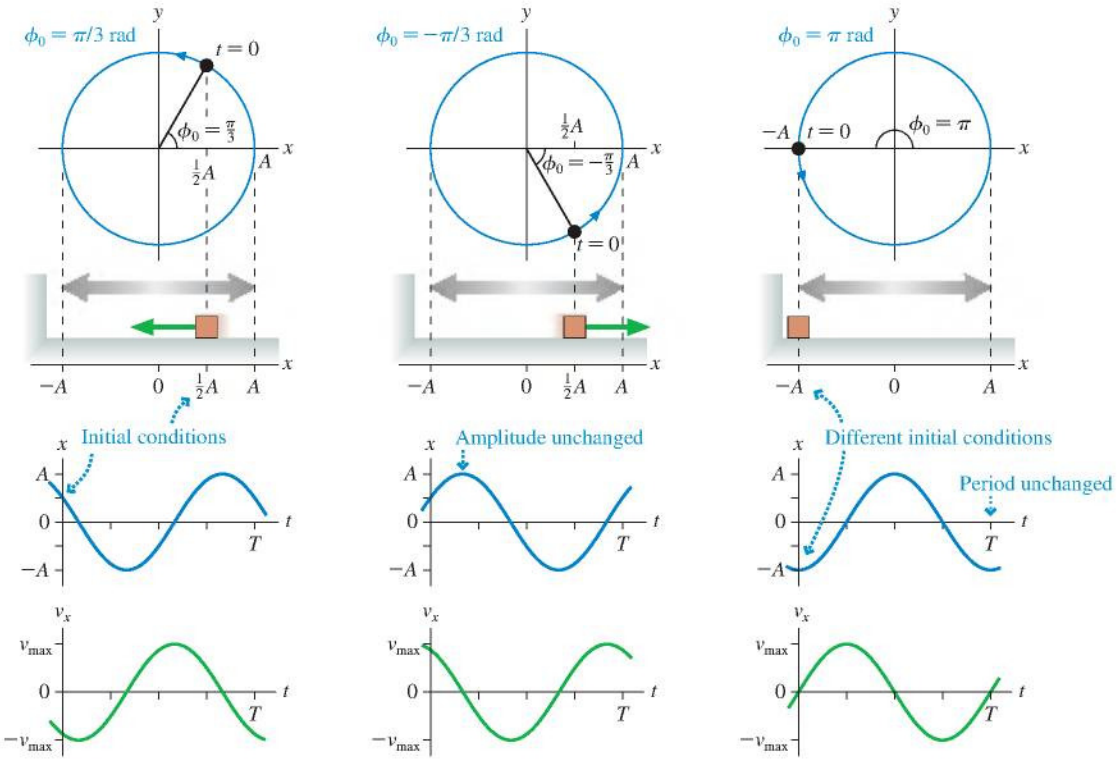


FIGURE 15.7 Different initial conditions are described by different values of the phase constant.



EXAMPLE 15.3 Using the initial conditions

An object on a spring oscillates with a period of 0.80 s and an amplitude of 10 cm. At $t = 0$ s, it is 5.0 cm to the left of equilibrium and moving to the left. What are its position and direction of motion at $t = 2.0$ s?

MODEL An object oscillating on a spring is in simple harmonic motion.

SOLVE We can find the phase constant ϕ_0 from the initial condition $x_0 = -5.0\text{ cm} = A\cos\phi_0$. This condition gives

$$\phi_0 = \cos^{-1}\left(\frac{x_0}{A}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \pm\tfrac{2}{3}\pi\text{ rad} = \pm 120^\circ$$

Because the oscillator is moving to the *left* at $t = 0$, it is in the upper half of the circular-motion diagram and must have a phase constant between 0 and π rad. Thus ϕ_0 is $\frac{2}{3}\pi$ rad. The angular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.80\text{ s}} = 7.85\text{ rad/s}$$

Thus the object's position at time $t = 2.0$ s is

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_0) \\ &= (10\text{ cm})\cos\big((7.85\text{ rad/s})(2.0\text{ s}) + \tfrac{2}{3}\pi\big) \\ &= (10\text{ cm})\cos(17.8\text{ rad}) = 5.0\text{ cm} \end{aligned}$$

The object is now 5.0 cm to the right of equilibrium. But which way is it moving? There are two ways to find out. The direct way is to calculate the velocity at $t = 2.0$ s:

$$v_x = -\omega A\sin(\omega t + \phi_0) = +68\text{ cm/s}$$

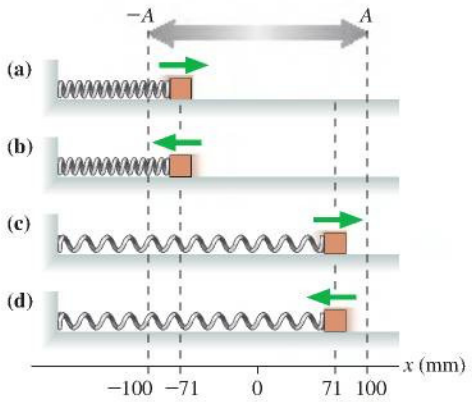
The velocity is positive, so the motion is to the right. Alternatively, we could note that the phase at $t = 2.0$ s is $\phi = 17.8$ rad. Dividing by π, you can see that

$$\phi = 17.8\text{ rad} = 5.67\pi\text{ rad} = (4\pi + 1.67\pi)\text{ rad}$$

The 4π rad represents two complete revolutions. The "extra" phase of 1.67π rad falls between π and 2π rad, so the particle in the circular-motion diagram is in the lower half of the circle and moving to the right.

NOTE The inverse-cosine function $\cos^{-1}$ is a *two-valued* function. Your calculator returns a single value, an angle between 0 rad and π rad. But the negative of this angle is also a solution. As Example 15.3 demonstrates, you must use additional information to choose between them.

STOP TO THINK 15.2 The figure shows four oscillators at $t = 0$. Which one has the phase constant $\phi_0 = \pi/4$ rad?



15.3 Energy in SHM

We've begun to develop the mathematical language of simple harmonic motion, but thus far we haven't included any physics. We've made no mention of the mass of the object or the spring constant of the spring. An energy analysis, using the tools of Chapter 10, is a good starting place.

FIGURE 15.8 shows an object oscillating on a spring, our prototype of simple harmonic motion. Now we'll specify that the object has mass m, the spring has spring constant k, and the motion takes place on a frictionless surface. You learned in Chapter 10 that the elastic potential energy when the object is at position x is $U_{Sp} = \frac{1}{2}k(\Delta x)^2$, where $\Delta x = x - x_{eq}$ is the displacement from the equilibrium position x_{eq}. In this chapter we'll always use a coordinate system in which $x_{eq} = 0$, making $\Delta x = x$. There's no chance for confusion with gravitational potential energy, so we can omit the subscript Sp and write the elastic potential energy as

$$U = \tfrac{1}{2}kx^2 \tag{15.17}$$

Thus the mechanical energy of an object oscillating on a spring is

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 \tag{15.18}$$

The lower portion of Figure 15.8 is an energy diagram, showing the parabolic potential-energy curve $U = \frac{1}{2}kx^2$ and the kinetic energy $K = E - U$. Recall that a particle oscillates between the *turning points* where the total energy line E crosses the potential-energy curve. The left turning point is at $x = -A$, and the right turning point is at $x = +A$. To go beyond these points would require a negative kinetic energy, which is physically impossible.

You can see that **the particle has purely potential energy at $x = \pm A$ and purely kinetic energy as it passes through the equilibrium point at $x = 0$.** At maximum displacement, with $x = \pm A$ and $v = 0$, the energy is

$$E(\text{at } x = \pm A) = U = \tfrac{1}{2}kA^2 \tag{15.19}$$

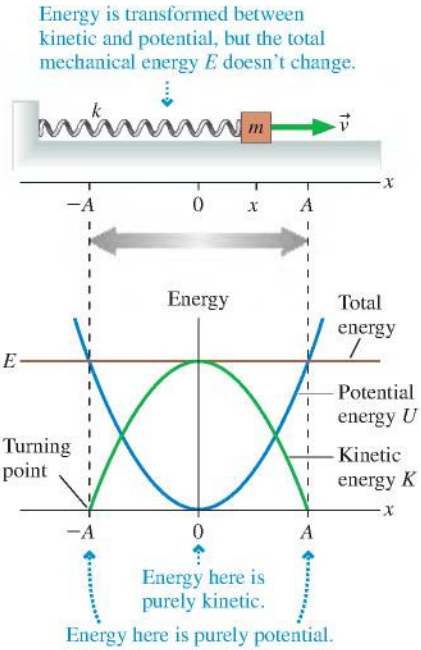
At $x = 0$, where $v = \pm v_{max}$, the energy is

$$E(\text{at } x = 0) = K = \tfrac{1}{2}m(v_{max})^2 \tag{15.20}$$

The system's mechanical energy is conserved because the surface is frictionless and there are no external forces, so the energy at maximum displacement and the energy at maximum speed, Equations 15.19 and 15.20, must be equal. That is

$$\tfrac{1}{2}m(v_{max})^2 = \tfrac{1}{2}kA^2 \tag{15.21}$$

FIGURE 15.8 Energy transformations during SHM.



Thus the maximum speed is related to the amplitude by

$$v_{max} = \sqrt{\frac{k}{m}}A \tag{15.22}$$

This is a relationship based on the physics of the situation.

Earlier, using kinematics, we found that

$$v_{max} = \frac{2\pi A}{T} = 2\pi f A = \omega A \tag{15.23}$$

Comparing Equations 15.22 and 15.23, we see that frequency and period of an oscillating spring are determined by the spring constant k and the object's mass m:

$$\omega = \sqrt{\frac{k}{m}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}} \tag{15.24}$$

These three expressions are really only one equation. They say the same thing, but each expresses it in slightly different terms.

Equations 15.24 tell us that the period and frequency are related to the object's mass m and the spring constant k. It is perhaps surprising, but **the period and frequency do not depend on the amplitude A.** A small oscillation and a large oscillation have the same period.

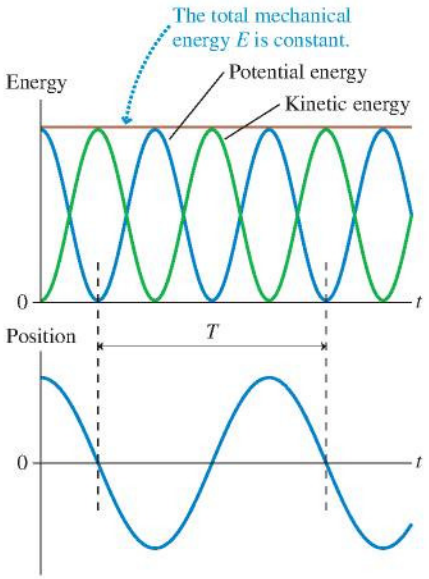


FIGURE 15.9 Kinetic energy, potential energy, and the total mechanical energy for simple harmonic motion.

Conservation of Energy

Because energy is conserved, we can combine Equations 15.18, 15.19, and 15.20 to write

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 = \tfrac{1}{2}m(v_{max})^2 \quad \text{(conservation of energy)} \tag{15.25}$$

Any pair of these expressions may be useful, depending on the known information. For example, you can use the amplitude A to find the speed at any point x by combining the first and second expressions for E. The speed v at position x is

$$v = \sqrt{\frac{k}{m}(A^2 - x^2)} = \omega\sqrt{A^2 - x^2} \tag{15.26}$$

FIGURE 15.9 shows graphically how the kinetic and potential energy change with time. They both oscillate but remain *positive* because x and v are squared. Energy is continuously being transformed back and forth between the kinetic energy of the moving block and the stored potential energy of the spring, but their sum remains constant. Notice that K and U both oscillate *twice* each period; make sure you understand why.

EXAMPLE 15.4 Using conservation of energy

A 500 g block on a spring is pulled a distance of 20 cm and released. The subsequent oscillations are measured to have a period of 0.80 s.

a. At what position or positions is the block's speed 1.0 m/s?

b. What is the spring constant?

MODEL The motion is SHM. Energy is conserved.

SOLVE a. The block starts from the point of maximum displacement, where $E = U = \frac{1}{2}kA^2$. At a later time, when the position is x and the speed is v, energy conservation requires

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2$$

Solving for x, we find

$$x = \sqrt{A^2 - \frac{mv^2}{k}} = \sqrt{A^2 - \left(\frac{v}{\omega}\right)^2}$$

where we used $k/m = \omega^2$ from Equation 15.24. The angular frequency is easily found from the period: $\omega = 2\pi/T = 7.85$ rad/s. Thus

$$x = \sqrt{(0.20\text{ m})^2 - \left(\frac{1.0\text{ m/s}}{7.85\text{ rad/s}}\right)^2} = \pm 0.15\text{ m} = \pm 15\text{ cm}$$

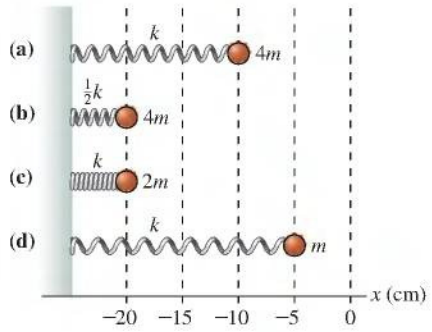
There are two positions because the block has this speed on either side of equilibrium.

b. Although part a did not require that we know the spring constant, it is straightforward to find from Equation 15.24:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$k = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2(0.50\text{ kg})}{(0.80\text{ s})^2} = 31\text{ N/m}$$

STOP TO THINK 15.3 The four springs shown here have been compressed from their equilibrium position at $x = 0$ cm. When released, the attached mass will start to oscillate. Rank in order, from highest to lowest, the maximum speeds of the masses.

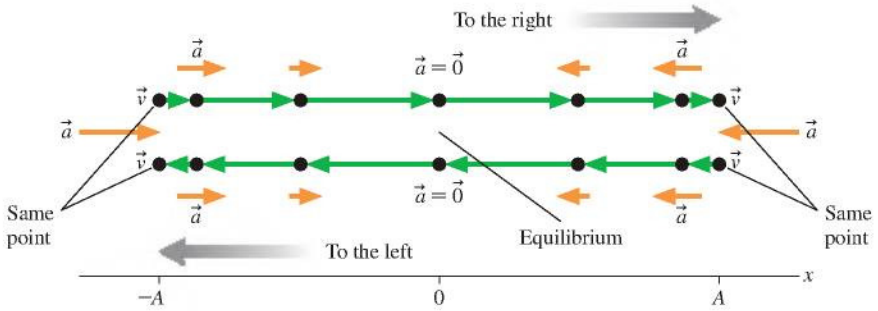


15.4 The Dynamics of SHM

Our analysis thus far has been based on the experimental observation that the oscillation of a spring "looks" sinusoidal. It's time to look at force and acceleration and to see that Newton's second law *predicts* sinusoidal motion.

A motion diagram will help us visualize the object's acceleration. **FIGURE 15.10** shows one cycle of the motion, separating motion to the left and motion to the right to make the diagram clear. As you can see, the object's velocity is large as it passes through the equilibrium point at $x = 0$, but $\vec{v}$ is *not changing* at that point. Acceleration measures the *change* of the velocity; hence $\vec{a} = \vec{0}$ at $x = 0$.

FIGURE 15.10 Motion diagram of simple harmonic motion. The left and right motions are separated vertically for clarity but really occur along the same line.



In contrast, the velocity is changing rapidly at the turning points. At the right turning point, $\vec{v}$ changes from a right-pointing vector to a left-pointing vector. Thus the acceleration $\vec{a}$ at the right turning point is large and *to the left*. In one-dimensional motion, the acceleration component a_x has a large *negative* value at the right turning point. Similarly, the acceleration $\vec{a}$ at the left turning point is large and *to the right*. Consequently, a_x has a large positive value at the left turning point.

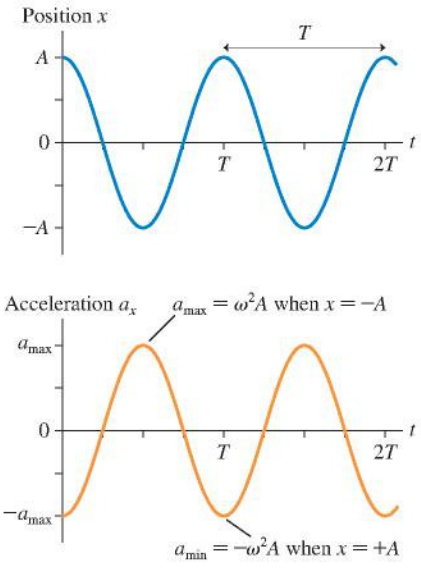
Our motion-diagram analysis suggests that the acceleration a_x is most positive when the displacement is most negative, most negative when the displacement is a maximum, and zero when $x = 0$. This is confirmed by taking the derivative of the velocity:

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(-\omega A \sin \omega t) = -\omega^2 A \cos \omega t \qquad (15.27)$$

then graphing it.

FIGURE 15.11 shows the position graph that we started with in Figure 15.3 and the corresponding acceleration graph. Comparing the two, you can see that the acceleration

FIGURE 15.11 Position and acceleration graphs for an oscillating spring. We've chosen $\phi_0 = 0$.

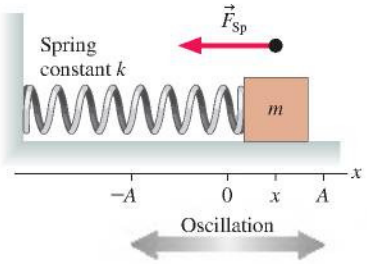


graph looks like an upside-down position graph. In fact, because $x = A\cos\omega t$, Equation 15.27 for the acceleration can be written

$$a_x = -\omega^2 x \tag{15.28}$$

That is, **the acceleration is proportional to the negative of the displacement.** The acceleration is, indeed, most positive when the displacement is most negative and is most negative when the displacement is most positive.

FIGURE 15.12 The prototype of simple harmonic motion: a mass oscillating on a horizontal spring without friction.



Recall that the acceleration is related to the net force by Newton's second law. Consider again our prototype mass on a spring, shown in FIGURE 15.12. This is the simplest possible oscillation, with no distractions due to friction or gravitational forces. We will assume the spring itself to be massless.

You learned in Chapter 9 that the spring force is given by Hooke's law:

$$(F_{Sp})_x = -k\,\Delta x \tag{15.29}$$

The minus sign indicates that the spring force is a **restoring force,** a force that always points back toward the equilibrium position. If we place the origin of the coordinate system at the equilibrium position, as we've done throughout this chapter, then $\Delta x = x$ and Hooke's law is simply $(F_{Sp})_x = -kx$.

The x-component of Newton's second law for the object attached to the spring is

$$(F_{net})_x = (F_{Sp})_x = -kx = ma_x \tag{15.30}$$

Equation 15.30 is easily rearranged to read

$$a_x = -\frac{k}{m}x \tag{15.31}$$

You can see that Equation 15.31 is identical to Equation 15.28 if the system oscillates with angular frequency $\omega = \sqrt{k/m}$. We previously found this expression for ω from an energy analysis. Our experimental observation that the acceleration is proportional to the *negative* of the displacement is exactly what Hooke's law would lead us to expect. That's the good news.

The bad news is that a_x is not a constant. As the object's position changes, so does the acceleration. Nearly all of our kinematic tools have been based on constant acceleration. We can't use those tools to analyze oscillations, so we must go back to the very definition of acceleration:

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

Acceleration is the second derivative of position with respect to time. If we use this definition in Equation 15.31, it becomes

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \text{(equation of motion for a mass on a spring)} \tag{15.32}$$

Equation 15.32, which is called the **equation of motion,** is a second-order differential equation. Unlike other equations we've dealt with, Equation 15.32 cannot be solved by direct integration. We'll need to take a different approach.

Solving the Equation of Motion

The solution to an algebraic equation such as $x^2 = 4$ is a number. The solution to a differential equation is a *function*. The x in Equation 15.32 is really $x(t)$, the position as a function of time. The solution to this equation is a function $x(t)$ whose second derivative is the function itself multiplied by $(-k/m)$.

One important property of differential equations that you will learn about in math is that the solutions are *unique*. That is, there is only *one* solution to Equation 15.32 that satisfies the initial conditions. If we were able to *guess* a solution, the uniqueness property would tell us that we had found the *only* solution. That might seem a rather

strange way to solve equations, but in fact differential equations are frequently solved by using your knowledge of what the solution needs to look like to guess an appropriate function. Let us give it a try!

We know from experimental evidence that the oscillatory motion of a spring appears to be sinusoidal. Let us *guess* that the solution to Equation 15.32 should have the functional form

$$x(t) = A\cos(\omega t + \phi_0) \tag{15.33}$$

where A, ω, and ϕ_0 are unspecified constants that we can adjust to any values that might be necessary to satisfy the differential equation.

If you were to guess that a solution to the algebraic equation $x^2 = 4$ is $x = 2$, you would verify your guess by substituting it into the original equation to see if it works. We need to do the same thing here: Substitute our guess for $x(t)$ into Equation 15.32 to see if, for an appropriate choice of the three constants, it works. To do so, we need the second derivative of $x(t)$. That is straightforward:

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_0) \\ \frac{dx}{dt} &= -\omega A\sin(\omega t + \phi_0) \\ \frac{d^2x}{dt^2} &= -\omega^2 A\cos(\omega t + \phi_0) \end{aligned} \tag{15.34}$$

If we now substitute the first and third of Equations 15.34 into Equation 15.32, we find

$$-\omega^2 A\cos(\omega t + \phi_0) = -\frac{k}{m}A\cos(\omega t + \phi_0) \tag{15.35}$$

Equation 15.35 will be true at all instants of time if and only if $\omega^2 = k/m$. There do not seem to be any restrictions on the two constants A and ϕ_0—they are determined by the initial conditions.

So we have found—by guessing!—that *the* solution to the equation of motion for a mass oscillating on a spring is

$$x(t) = A\cos(\omega t + \phi_0) \tag{15.36}$$

where the angular frequency

$$\omega = 2\pi f = \sqrt{\frac{k}{m}} \tag{15.37}$$

is determined by the mass and the spring constant.

NOTE Once again we see that the oscillation frequency is independent of the amplitude A.

Equations 15.36 and 15.37 seem somewhat anticlimactic because we've been using these results for the last several pages. But keep in mind that we had been *assuming* $x = A\cos\omega t$ simply because the experimental observations "looked" like a cosine function. We've now justified that assumption by showing that Equation 15.36 really is the solution to Newton's second law for a mass on a spring. **The *theory* of oscillation, based on Hooke's law for a spring and Newton's second law, is in good agreement with the experimental observations.**

An optical technique called *interferometry* reveals the bell-like vibrations of a wine glass.

EXAMPLE 15.5 | **Analyzing an oscillator**

At $t = 0$ s, a 500 g block oscillating on a spring is observed moving to the right at $x = 15$ cm. It reaches a maximum displacement of 25 cm at $t = 0.30$ s.

a. Draw a position-versus-time graph for one cycle of the motion.

b. What is the maximum force on the block, and what is the first time at which this occurs?

Continued

MODEL The motion is simple harmonic motion.

SOLVE a. The position equation of the block is $x(t) = A\cos(\omega t + \phi_0)$. We know that the amplitude is $A = 0.25$ m and that $x_0 = 0.15$ m. From these two pieces of information we obtain the phase constant:

$$\phi_0 = \cos^{-1}\left(\frac{x_0}{A}\right) = \cos^{-1}(0.60) = \pm 0.927 \text{ rad}$$

The object is initially moving to the right, which tells us that the phase constant must be between $-\pi$ and 0 rad. Thus $\phi_0 = -0.927$ rad. The block reaches its maximum displacement $x_{max} = A$ at time $t = 0.30$ s. At that instant of time

$$x_{max} = A = A\cos(\omega t + \phi_0)$$

This equation can be true only if $\cos(\omega t + \phi_0) = 1$, which requires $\omega t + \phi_0 = 0$. Thus

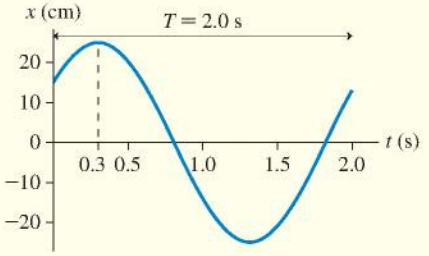
$$\omega = \frac{-\phi_0}{t} = \frac{-(-0.927 \text{ rad})}{0.30 \text{ s}} = 3.09 \text{ rad/s}$$

Now that we know ω, it is straightforward to compute the period:

$$T = \frac{2\pi}{\omega} = 2.0 \text{ s}$$

FIGURE 15.13 graphs $x(t) = (25 \text{ cm})\cos(3.09t - 0.927)$, where t is in s, from $t = 0$ s to $t = 2.0$ s.

FIGURE 15.13 Position-versus-time graph for the oscillator of Example 15.5.



b. We found the acceleration of SHM to be $a_x = -\omega^2 x$, so the force on the block at position x is $F_x = -m\omega^2 x$. The force will be a maximum, $F_{max} = m\omega^2 A$, when x reaches its minimum displacement $x = -A$. For the block,

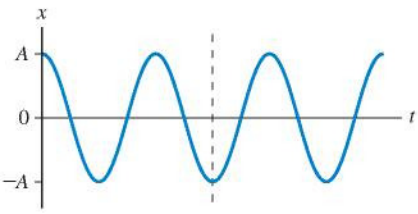
$$F_{max} = (0.50 \text{ kg})(3.09 \text{ rad/s})^2(0.25 \text{ m}) = 1.2 \text{ N}$$

This occurs exactly half a period (1.0 s) after the block reaches its maximum displacement, thus at $t = 1.3$ s.

ASSESS A 2 s period is a modest oscillation, so we don't expect the block's acceleration to be extreme. A maximum force of 1.2 N on a 0.5 kg block causes a maximum acceleration of 2.4 m/s^2, which seems reasonable.

STOP TO THINK 15.4 This is the position graph of a mass on a spring. What can you say about the velocity and the force at the instant indicated by the dashed line?

a. Velocity positive; force to the right.
b. Velocity negative; force to the right.
c. Velocity zero; force to the right.
d. Velocity positive; force to the left.
e. Velocity negative; force to the left.
f. Velocity zero; force to the left.
g. Velocity and force both zero.



15.5 Vertical Oscillations

We have focused our analysis on a horizontally oscillating spring. But the typical demonstration you'll see in class is a mass bobbing up and down on a spring hung vertically from a support. Is it safe to assume that a vertical oscillation has the same mathematical description as a horizontal oscillation? Or does the additional force of gravity change the motion? Let us look at this more carefully.

FIGURE 15.14 Gravity stretches the spring.

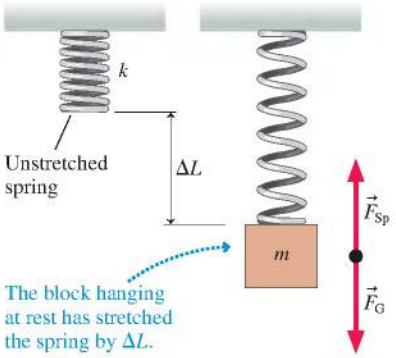


FIGURE 15.14 shows a block of mass m hanging from a spring of spring constant k. An important fact to notice is that the equilibrium position of the block is *not* where the spring is at its unstretched length. At the equilibrium position of the block, where it hangs motionless, the spring has stretched by ΔL.

Finding ΔL is an equilibrium problem in which the upward spring force balances the downward gravitational force on the block. The y-component of the spring force is given by Hooke's law:

$$(F_{Sp})_y = -k\,\Delta y = +k\,\Delta L \qquad (15.38)$$

Equation 15.38 makes a distinction between ΔL, which is simply a *distance* and is a positive number, and the displacement Δy. The block is displaced downward, so $\Delta y = -\Delta L$. Newton's first law for the block in equilibrium is

$$(F_{\text{net}})_y = (F_{\text{Sp}})_y + (F_{\text{G}})_y = k\,\Delta L - mg = 0 \tag{15.39}$$

from which we can find

$$\Delta L = \frac{mg}{k} \tag{15.40}$$

This is the distance the spring stretches when the block is attached to it.

Let the block oscillate around this equilibrium position, as shown in FIGURE 15.15. We've now placed the origin of the y-axis at the block's equilibrium position in order to be consistent with our analyses of oscillations throughout this chapter. If the block moves upward, as the figure shows, the spring gets shorter compared to its equilibrium length, but the spring is still *stretched* compared to its unstretched length in Figure 15.14. When the block is at position y, the spring is stretched by an amount $\Delta L - y$ and hence exerts an *upward* spring force $F_{\text{Sp}} = k(\Delta L - y)$. The net force on the block at this point is

$$(F_{\text{net}})_y = (F_{\text{Sp}})_y + (F_{\text{G}})_y = k(\Delta L - y) - mg = (k\,\Delta L - mg) - ky \tag{15.41}$$

But $k\,\Delta L - mg$ is zero, from Equation 15.40, so the net force on the block is simply

$$(F_{\text{net}})_y = -ky \tag{15.42}$$

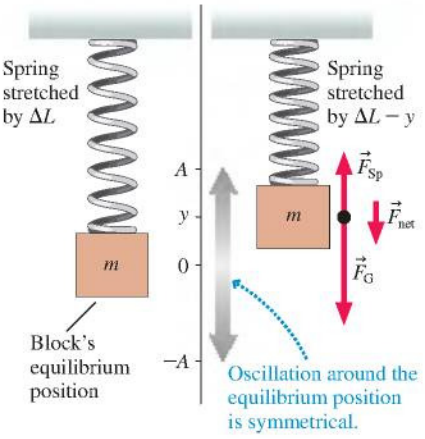
Equation 15.42 for vertical oscillations is *exactly* the same as Equation 15.30 for horizontal oscillations, where we found $(F_{\text{net}})_x = -kx$. That is, the restoring force for vertical oscillations is identical to the restoring force for horizontal oscillations. The role of gravity is to determine where the equilibrium position is, but it doesn't affect the oscillatory motion around the equilibrium position.

Because the net force is the same, Newton's second law has exactly the same oscillatory solution:

$$y(t) = A\cos(\omega t + \phi_0) \tag{15.43}$$

with, again, $\omega = \sqrt{k/m}$. **The vertical oscillations of a mass on a spring are the same simple harmonic motion as those of a block on a horizontal spring.** This is an important finding because it was not obvious that the motion would still be simple harmonic motion when gravity was included.

FIGURE 15.15 The block oscillates around the equilibrium position.



EXAMPLE 15.6 Bungee oscillations

An 83 kg student hangs from a bungee cord with spring constant 270 N/m. The student is pulled down to a point where the cord is 5.0 m longer than its unstretched length, then released. Where is the student, and what is his velocity 2.0 s later?

MODEL A bungee cord can be modeled as a spring. Vertical oscillations on the bungee cord are SHM.

VISUALIZE FIGURE 15.16 shows the situation.

SOLVE Although the cord is stretched by 5.0 m when the student is released, this is *not* the amplitude of the oscillation. Oscillations occur around the equilibrium position, so we have to begin by finding the equilibrium point where the student hangs motionless. The cord stretch at equilibrium is given by Equation 15.40:

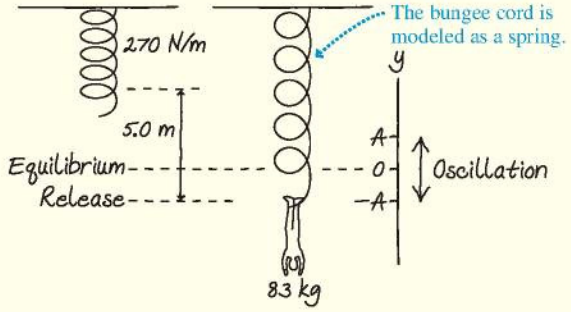
$$\Delta L = \frac{mg}{k} = 3.0\text{ m}$$

Stretching the cord 5.0 m pulls the student 2.0 m below the equilibrium point, so $A = 2.0$ m. That is, the student oscillates with amplitude $A = 2.0$ m about a point 3.0 m beneath the bungee cord's original end point. The student's position as a function of time, as measured from the equilibrium position, is

$$y(t) = (2.0\text{ m})\cos(\omega t + \phi_0)$$

where $\omega = \sqrt{k/m} = 1.80$ rad/s.

FIGURE 15.16 A student on a bungee cord oscillates about the equilibrium position.



Continued

The initial condition

$$y_0 = A\cos\phi_0 = -A$$

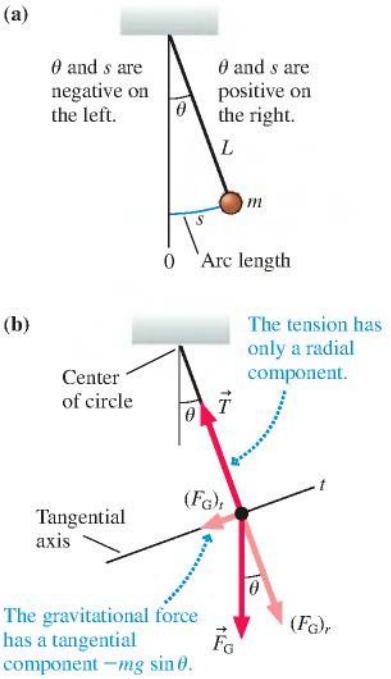
requires the phase constant to be $\phi_0 = \pi$ rad. At $t = 2.0$ s the student's position and velocity are

$$y = (2.0\text{ m})\cos\big((1.80\text{ rad/s})(2.0\text{ s}) + \pi\text{ rad}\big) = 1.8\text{ m}$$
$$v_y = -\omega A\sin(\omega t + \phi_0) = -1.6\text{ m/s}$$

The student is 1.8 m *above* the equilibrium position, or 1.2 m *below* the original end of the cord. Because his velocity is negative, he's passed through the highest point and is heading down.

15.6 The Pendulum

FIGURE 15.17 Pendulum motion



Now let's look at another very common oscillator: a pendulum. **FIGURE 15.17a** shows a mass m attached to a string of length L and free to swing back and forth. The pendulum's position can be described by the arc of length s, which is zero when the pendulum hangs straight down. Because angles are measured ccw, s and θ are positive when the pendulum is to the right of center, negative when it is to the left.

Two forces are acting on the mass: the string tension $\vec{T}$ and gravity $\vec{F}_G$. As we did with circular motion, it will be useful to divide the forces into tangential components, parallel to the motion, and radial components parallel to the string. These are shown on the free-body diagram of **FIGURE 15.17b**.

Newton's second law for the tangential component, parallel to the motion, is

$$(F_{net})_t = \sum F_t = (F_G)_t = -mg\sin\theta = ma_t \qquad (15.44)$$

Using $a_t = d^2s/dt^2$ for acceleration "around" the circle, and noting that the mass cancels, we can write Equation 15.44 as

$$\frac{d^2s}{dt^2} = -g\sin\theta \qquad (15.45)$$

This is the equation of motion for an oscillating pendulum. The sine function makes this equation more complicated than the equation of motion for an oscillating spring.

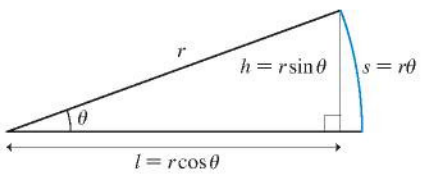
The Small-Angle Approximation

Suppose we restrict the pendulum's oscillations to *small angles* of less than about 10°. This restriction allows us to make use of an interesting and important piece of geometry.

FIGURE 15.18 shows an angle θ and a circular arc of length $s = r\theta$. A right triangle has been constructed by dropping a perpendicular from the top of the arc to the axis. The height of the triangle is $h = r\sin\theta$. Suppose that the angle θ is "small," which, in practice, means $\theta \ll 1$ rad. In that case there is very little difference between h and s. If $h \approx s$, then $r\sin\theta \approx r\theta$. It follows that

$$\sin\theta \approx \theta \quad \text{if} \quad \theta \ll 1\text{ rad}$$

FIGURE 15.18 The geometrical basis of the small-angle approximation.



The result that $\sin\theta \approx \theta$ for small angles is called the **small-angle approximation.** We can similarly note that $l \approx r$ for small angles. Because $l = r\cos\theta$, it follows that

$$\cos\theta \approx 1 \quad \text{if} \quad \theta \ll 1\text{ rad}$$

Finally, we can take the ratio of sine and cosine to find $\tan\theta \approx \sin\theta \approx \theta$. We will have other occasions to use the small-angle approximation throughout the remainder of this text.

NOTE The small-angle approximation is valid *only* if angle θ is in radians!

How small does θ have to be to justify using the small-angle approximation? It's easy to use your calculator to find that the small-angle approximation is good to three significant figures, an error of $\leq 0.1\%$, up to angles of ≈ 0.10 rad ($\approx 5°$). In practice, we will use the approximation up to about 10°, but for angles any larger it rapidly loses validity and produces unacceptable results.

If we restrict the pendulum to $\theta < 10°$, we can use $\sin\theta \approx \theta$. In that case, Equation 15.44 for the net force on the mass is

$$(F_{\text{net}})_t = -mg\sin\theta \approx -mg\theta = -\frac{mg}{L}s$$

where, in the last step, we used the fact that angle θ is related to the arc length by $\theta = s/L$. Then the equation of motion becomes

$$\frac{d^2s}{dt^2} = \frac{(F_{\text{net}})_t}{m} = -\frac{g}{L}s \qquad (15.46)$$

This is *exactly* the same as Equation 15.32 for a mass oscillating on a spring. The names are different, with x replaced by s and k/m by g/L, but that does not make it a different equation.

Because we know the solution to the spring problem, we can immediately write the solution to the pendulum problem just by changing variables and constants:

$$s(t) = A\cos(\omega t + \phi_0) \quad \text{or} \quad \theta(t) = \theta_{\max}\cos(\omega t + \phi_0) \qquad (15.47)$$

The angular frequency

$$\omega = 2\pi f = \sqrt{\frac{g}{L}} \qquad (15.48)$$

is determined by the length of the string. The pendulum is interesting in that **the frequency, and hence the period, is independent of the mass.** It depends only on the length of the pendulum. The amplitude A and the phase constant ϕ_0 are determined by the initial conditions, just as they were for an oscillating spring.

EXAMPLE 15.7 The maximum angle of a pendulum

A 300 g mass on a 30-cm-long string oscillates as a pendulum. It has a speed of 0.25 m/s as it passes through the lowest point. What maximum angle does the pendulum reach?

MODEL Assume that the angle remains small, in which case the motion is simple harmonic motion.

SOLVE The angular frequency of the pendulum is

$$\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8\ \text{m/s}^2}{0.30\ \text{m}}} = 5.72\ \text{rad/s}$$

The speed at the lowest point is $v_{\max} = \omega A$, so the amplitude is

$$A = s_{\max} = \frac{v_{\max}}{\omega} = \frac{0.25\ \text{m/s}}{5.72\ \text{rad/s}} = 0.0437\ \text{m}$$

The maximum angle, at the maximum arc length $s_{\max}$, is

$$\theta_{\max} = \frac{s_{\max}}{L} = \frac{0.0437\ \text{m}}{0.30\ \text{m}} = 0.146\ \text{rad} = 8.3°$$

ASSESS Because the maximum angle is less than 10°, our analysis based on the small-angle approximation is reasonable.

EXAMPLE 15.8 The gravimeter

Deposits of minerals and ore can alter the local value of the free-fall acceleration because they tend to be denser than surrounding rocks. Geologists use a *gravimeter*—an instrument that accurately measures the local free-fall acceleration—to search for ore deposits. One of the simplest gravimeters is a pendulum. To achieve the highest accuracy, a stopwatch is used to time 100 oscillations of a pendulum of different lengths. At one location in the field, a geologist makes the following measurements:

Length (m)	Time (s)
0.500	141.7
1.000	200.6
1.500	245.8
2.000	283.5

What is the local value of g?

MODEL Assume the oscillation angle is small, in which case the motion is simple harmonic motion with a period independent of the mass of the pendulum. Because the data are known to four significant figures (± 1 mm on the length and ± 0.1 s on the timing, both of which are easily achievable), we expect to determine g to four significant figures.

SOLVE From Equation 15.48, using $f = 1/T$, we find

$$T^2 = \left(2\pi\sqrt{\frac{L}{g}}\right)^2 = \frac{4\pi^2}{g}L$$

That is, the square of a pendulum's period is proportional to its length. Consequently, a graph of T^2 versus L should be a straight line passing through the origin with slope $4\pi^2/g$. We can use the

Continued

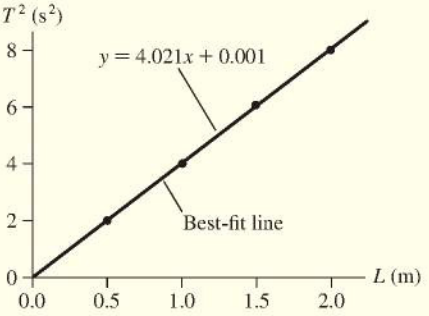
experimentally measured slope to determine g. FIGURE 15.19 is a graph of the data, with the period found by dividing the measured time by 100.

As expected, the graph is a straight line passing through the origin. The slope of the best-fit line is 4.021 s^2/m. Consequently,

$$g = \frac{4\pi^2}{\text{slope}} = \frac{4\pi^2}{4.021\ s^2/m} = 9.818\ m/s^2$$

ASSESS The fact that the graph is linear and passes through the origin confirms our model of the situation. Had this *not* been the case, we would have had to conclude either that our model of the pendulum as a simple, small-angle pendulum was not valid or that our measurements were bad. This is an important reason for having multiple data points rather than using only one length.

FIGURE 15.19 Graph of the square of the pendulum's period versus its length.



The Simple-Harmonic-Motion Model

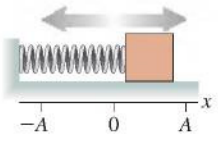
You can begin to see how, in a sense, we have solved *all* simple-harmonic-motion problems once we have solved the problem of the horizontal spring. The restoring force of a spring, $F_{Sp} = -kx$, is directly proportional to the displacement x from equilibrium. The pendulum's restoring force, in the small-angle approximation, is directly proportional to the displacement s. A restoring force that is directly proportional to the displacement from equilibrium is called a **linear restoring force.** For *any* linear restoring force, the equation of motion is identical to the spring equation (other than perhaps using different symbols). Consequently, **any system with a linear restoring force will undergo simple harmonic motion around the equilibrium position.**

This is why an oscillating spring is the prototype of SHM. Everything that we learn about an oscillating spring can be applied to the oscillations of any other linear restoring force, ranging from the vibration of airplane wings to the motion of electrons in electric circuits.

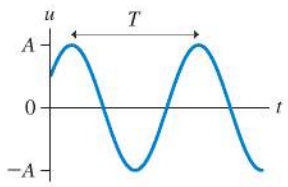
MODEL 15.1

Simple harmonic motion

For any system with a restoring force that's linear or can be well approximated as linear.



- Motion is SHM around the equilibrium position.
- Frequency and period are independent of the amplitude.
- Mathematically:
  - For an appropriate position variable u, the equation of motion can be written



$$d^2u/dt^2 = -Cu$$

    where C is a collection of constants.
  - The angular frequency is $\omega = \sqrt{C}$.
  - The position and velocity are

$$u = A\cos(\omega t + \phi_0) \quad v_u = -v_{max}\sin(\omega t + \phi_0)$$

    where A and ϕ_0 are determined by the initial conditions.
  - Mechanical energy is conserved.
- Limitations: Model fails if the restoring force deviates significantly from linear.

Exercise 22

The Physical Pendulum

A mass on a string is often called a *simple pendulum*. But you can also make a pendulum from any solid object that swings back and forth on a pivot under the influence of gravity. This is called a *physical pendulum*.

FIGURE 15.20 shows a physical pendulum of mass M for which the distance between the pivot and the center of mass is l. The moment arm of the gravitational force acting at the center of mass is $d = l\sin\theta$, so the gravitational torque is

$$\tau = -Mgd = -Mgl\sin\theta$$

The torque is negative because, for positive θ, it's causing a clockwise rotation. If we restrict the angle to being small ($\theta < 10°$), as we did for the simple pendulum, we can use the small-angle approximation to write

$$\tau = -Mgl\theta \tag{15.49}$$

Gravity exerts a linear restoring torque on the pendulum—that is, the torque is directly proportional to the angular displacement θ—so we expect the physical pendulum to undergo SHM.

From Chapter 12, Newton's second law for rotational motion is

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{\tau}{I}$$

where I is the object's moment of inertia about the pivot point. Using Equation 15.49 for the torque, we find

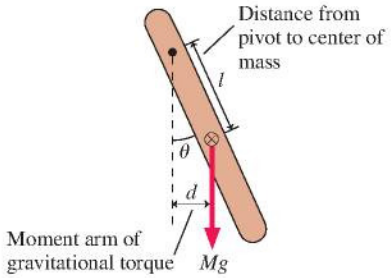
$$\frac{d^2\theta}{dt^2} = \frac{-Mgl}{I}\theta \tag{15.50}$$

The equation of motion is of the form $d^2\theta/dt^2 = -C\theta$, so the model for simple harmonic motion tells us that the motion is SHM with angular frequency

$$\omega = 2\pi f = \sqrt{\frac{Mgl}{I}} \tag{15.51}$$

It appears that the frequency depends on the mass of the pendulum, but recall that the moment of inertia is directly proportional to M. Thus M cancels and the frequency of a physical pendulum, like that of a simple pendulum, is independent of mass.

FIGURE 15.20 A physical pendulum.



EXAMPLE 15.9 A swinging leg as a pendulum

A student in a biomechanics lab measures the length of his leg, from hip to heel, to be 0.90 m. What is the frequency of the pendulum motion of the student's leg? What is the period?

MODEL We can model a human leg reasonably well as a rod of uniform cross section, pivoted at one end (the hip) to form a physical pendulum. For small-angle oscillations it will undergo SHM. The center of mass of a uniform leg is at the midpoint, so $l = L/2$.

SOLVE The moment of inertia of a rod pivoted about one end is $I = \frac{1}{3}ML^2$, so the pendulum frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{Mgl}{I}} = \frac{1}{2\pi}\sqrt{\frac{Mg(L/2)}{ML^2/3}} = \frac{1}{2\pi}\sqrt{\frac{3g}{2L}} = 0.64\text{ Hz}$$

The corresponding period is $T = 1/f = 1.6$ s. Notice that we didn't need to know the mass.

ASSESS As you walk, your legs do swing as physical pendulums as you bring them forward. The frequency is fixed by the length of your legs and their distribution of mass; it doesn't depend on amplitude. Consequently, you don't increase your walking speed by taking more rapid steps—changing the frequency is difficult. You simply take longer strides, changing the amplitude but not the frequency.

STOP TO THINK 15.5 One person swings on a swing and finds that the period is 3.0 s. A second person of equal mass joins him. With two people swinging, the period is

a. 6.0 s
b. >3.0 s but not necessarily 6.0 s
c. 3.0 s
d. <3.0 s but not necessarily 1.5 s
e. 1.5 s
f. Can't tell without knowing the length

The shock absorbers in cars and trucks are heavily damped springs. The vehicle's vertical motion, after hitting a rock or a pothole, is a damped oscillation.

15.7 Damped Oscillations

A pendulum left to itself gradually slows down and stops. The sound of a ringing bell gradually dies away. All real oscillators do run down—some very slowly but others quite quickly—as friction or other dissipative forces transform their mechanical energy into the thermal energy of the oscillator and its environment. An oscillation that runs down and stops is called a **damped oscillation.**

There are many possible reasons for the dissipation of energy, such as air resistance, friction, and internal forces within a metal spring as it flexes. The forces involved in dissipation are complex, but a simple *linear drag* model gives a quite accurate description of most damped oscillations. That is, we'll assume a drag force that depends linearly on the velocity as

$$\vec{F}_{\text{drag}} = -b\vec{v} \quad \text{(model of the drag force)} \tag{15.52}$$

where the minus sign is the mathematical statement that the force is always opposite in direction to the velocity in order to slow the object.

The **damping constant** b depends in a complicated way on the shape of the object *and* on the viscosity of the air or other medium in which the particle moves. The damping constant plays the same role in our model of drag that the coefficient of friction does in our model of friction.

The units of b need to be such that they will give units of force when multiplied by units of velocity. As you can confirm, these units are kg/s. A value $b = 0$ kg/s corresponds to the limiting case of no resistance, in which case the mechanical energy is conserved. A typical value of b for a spring or a pendulum in air is ≤ 0.10 kg/s. Objects moving in a liquid can have significantly larger values of b.

FIGURE 15.21 An oscillating mass in the presence of a drag force.

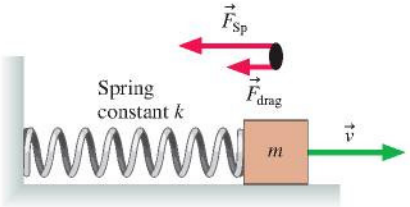


FIGURE 15.21 shows a mass oscillating on a spring in the presence of a drag force. With the drag included, Newton's second law is

$$(F_{\text{net}})_x = (F_{\text{Sp}})_x + (F_{\text{drag}})_x = -kx - bv_x = ma_x \tag{15.53}$$

Using $v_x = dx/dt$ and $a_x = d^2x/dt^2$, we can write Equation 15.53 as

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \tag{15.54}$$

Equation 15.54 is the equation of motion of a damped oscillator. If you compare it to Equation 15.32, the equation of motion for a block on a frictionless surface, you'll see that it differs by the inclusion of the term involving dx/dt.

Equation 15.54 is another second-order differential equation. We will simply assert (and, as a homework problem, you can confirm) that the solution is

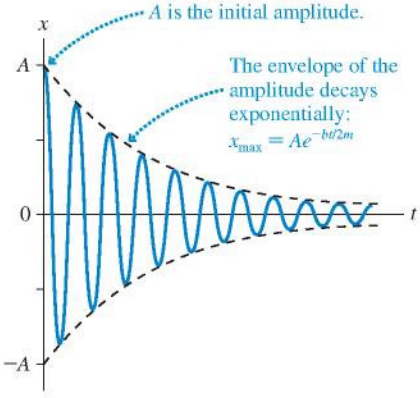
$$x(t) = Ae^{-bt/2m}\cos(\omega t + \phi_0) \quad \text{(damped oscillator)} \tag{15.55}$$

where the angular frequency is given by

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}} \tag{15.56}$$

Here $\omega_0 = \sqrt{k/m}$ is the angular frequency of an undamped oscillator ($b = 0$). The constant e is the base of natural logarithms, so $e^{-bt/2m}$ is an *exponential function*. Because $e^0 = 1$, Equation 15.55 reduces to our previous $x(t) = A\cos(\omega t + \phi_0)$ when $b = 0$. This makes sense and gives us confidence in Equation 15.55.

FIGURE 15.22 Position-versus-time graph for a lightly damped oscillator.



Lightly Damped Oscillators

A *lightly damped* oscillator, which oscillates many times before stopping, is one for which $b/2m \ll \omega_0$. In that case, $\omega \approx \omega_0$ is a good approximation. That is, light damping does not affect the oscillation frequency.

FIGURE 15.22 is a graph of the position $x(t)$ for a lightly damped oscillator, as given by Equation 15.55. To keep things simple, we've assumed that the phase constant is

zero. Notice that the term $Ae^{-bt/2m}$, which is shown by the dashed line, acts as a slowly varying amplitude:

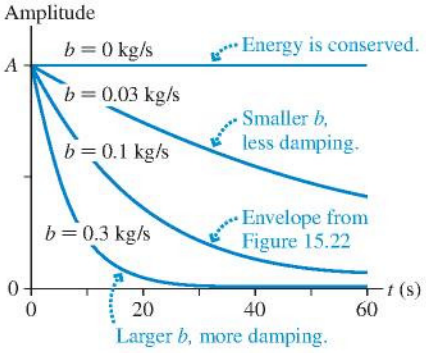
$$x_{max}(t) = Ae^{-bt/2m} \tag{15.57}$$

where A is the *initial* amplitude, at $t = 0$. The oscillation keeps bumping up against this line, slowly dying out with time.

A slowly changing line that provides a border to a rapid oscillation is called the **envelope** of the oscillations. In this case, the oscillations have an *exponentially decaying envelope*. Make sure you study Figure 15.22 long enough to see how both the oscillations and the decaying amplitude are related to Equation 15.55.

Changing the amount of damping, by changing the value of b, affects how quickly the oscillations decay. FIGURE 15.23 shows just the envelope $x_{max}(t)$ for several oscillators that are identical except for the value of the damping constant b. (You need to imagine a rapid oscillation within each envelope, as in Figure 15.22.) Increasing b causes the oscillations to damp more quickly, while decreasing b makes them last longer.

FIGURE 15.23 Oscillation envelopes for several values of b, mass 1.0 kg.



MATHEMATICAL ASIDE

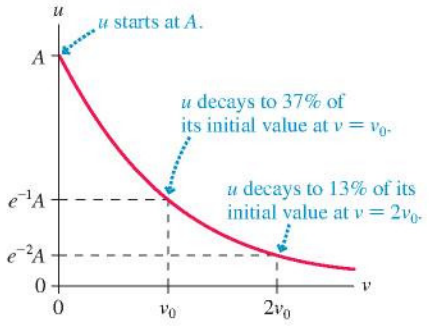
Exponential decay

Exponential decay occurs in a vast number of physical systems of importance in science and engineering. Mechanical vibrations, electric circuits, and nuclear radioactivity all exhibit exponential decay.

The mathematical analysis of physical systems frequently leads to solutions of the form

$$u = Ae^{-v/v_0} = A\exp(-v/v_0)$$

where exp is the *exponential function*. The number $e = 2.71828\ldots$ is the base of natural logarithms in the same way that 10 is the base of ordinary logarithms.



A graph of u illustrates what we mean by exponential decay. It starts with $u = A$ at $v = 0$ (because $e^0 = 1$) and then steadily decays, asymptotically approaching zero. The quantity v_0 is called the *decay constant*. When $v = v_0$, $u = e^{-1}A = 0.37A$. When $v = 2v_0$, $u = e^{-2}A = 0.13A$.

The decay constant v_0 must have the same units as v. If v represents position, then v_0 is a length; if v represents time, then v_0 is a time interval. In a specific situation, v_0 is often called the *decay length* or the *decay time*. It is the length or time in which the quantity decays to 37% of its initial value.

No matter what the process is or what u represents, **a quantity that decays exponentially decays to 37% of its initial value when one decay constant has passed.** Thus exponential decay is a universal behavior. The decay curve always looks exactly like the figure shown here. Once you've learned the properties of exponential decay, you'll immediately know how to apply this knowledge to a new situation.

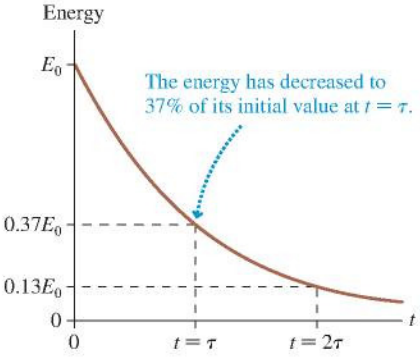
The mechanical energy of a damped oscillator is *not* conserved because of the drag force. We previously found the energy of an undamped oscillator to be $E = \frac{1}{2}kA^2$. This is still valid for a lightly damped oscillator if we replace A with the slowly decaying amplitude x_{max}. Thus

$$E(t) = \tfrac{1}{2}k(x_{max})^2 = \tfrac{1}{2}k(Ae^{-bt/2m})^2 = \tfrac{1}{2}kA^2e^{-bt/m} \tag{15.58}$$

Here A is the initial amplitude, so $\frac{1}{2}kA^2$ is the initial energy, which we call E_0. Let's define the **time constant** τ (also called the *decay constant* or the *decay time*) to be

$$\tau = \frac{m}{b} \tag{15.59}$$

FIGURE 15.24 Energy decay of a lightly damped oscillator.



Because b has units of kg/s, τ has units of seconds. With this, we can write the energy decay as

$$E(t) = E_0 e^{-t/\tau} \tag{15.60}$$

In other words, **a lightly damped oscillator's mechanical energy decays exponentially with time constant τ.**

As **FIGURE 15.24** shows, the time constant is the amount of time needed for the energy to decay to e^{-1}, or 37%, of its initial value. We say that the time constant τ measures the "characteristic time" during which the energy of the oscillation is dissipated. Roughly two-thirds of the initial energy is gone after one time constant has elapsed, and nearly 90% has dissipated after two time constants have gone by.

For practical purposes, we can speak of the time constant as the *lifetime* of the oscillation—about how long it lasts. Mathematically, there is never a time when the oscillation is "over." The decay approaches zero asymptotically, but it never gets there in any finite time. The best we can do is define a characteristic time when the motion is "almost over," and that is what the time constant τ does.

EXAMPLE 15.10 A damped pendulum

A 500 g mass swings on a 60-cm-string as a pendulum. The amplitude is observed to decay to half its initial value after 35 oscillations.

a. What is the time constant for this oscillator?

b. At what time will the *energy* have decayed to half its initial value?

MODEL The motion is a damped oscillation.

SOLVE a. The initial amplitude at $t = 0$ is $x_{max} = A$. After 35 oscillations the amplitude is $x_{max} = \frac{1}{2}A$. The period of the pendulum is

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{0.60\text{ m}}{9.8\text{ m/s}^2}} = 1.55\text{ s}$$

so 35 oscillations have occurred at $t = 54.2$ s.

The amplitude of oscillation at time t is given by Equation 15.57: $x_{max}(t) = Ae^{-bt/2m} = Ae^{-t/2\tau}$. In this case,

$$\tfrac{1}{2}A = Ae^{-(54.2\text{ s})/2\tau}$$

Notice that we do not need to know A itself because it cancels out. To solve for τ, we take the natural logarithm of both sides of the equation:

$$\ln\left(\tfrac{1}{2}\right) = -\ln 2 = \ln e^{-(54.2\text{ s})/2\tau} = -\frac{54.2\text{ s}}{2\tau}$$

This is easily rearranged to give

$$\tau = \frac{54.2\text{ s}}{2\ln 2} = 39\text{ s}$$

If desired, we could now determine the damping constant to be $b = m/\tau = 0.013$ kg/s.

b. The energy at time t is given by

$$E(t) = E_0 e^{-t/\tau}$$

The time at which an exponential decay is reduced to $\frac{1}{2}E_0$, half its initial value, has a special name. It is called the **half-life** and given the symbol $t_{1/2}$. The concept of the half-life is widely used in applications such as radioactive decay. To relate $t_{1/2}$ to τ, we first write

$$E(\text{at } t = t_{1/2}) = \tfrac{1}{2}E_0 = E_0 e^{-t_{1/2}/\tau}$$

The E_0 cancels, giving

$$\tfrac{1}{2} = e^{-t_{1/2}/\tau}$$

Again, we take the natural logarithm of both sides:

$$\ln\left(\tfrac{1}{2}\right) = -\ln 2 = \ln e^{-t_{1/2}/\tau} = -t_{1/2}/\tau$$

Finally, we solve for $t_{1/2}$:

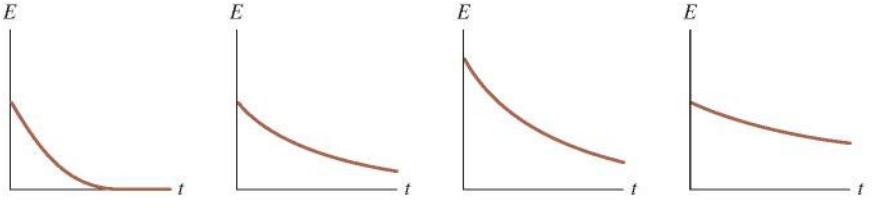
$$t_{1/2} = \tau \ln 2 = 0.693\tau$$

This result that $t_{1/2}$ is 69% of τ is valid for any exponential decay. In this particular problem, half the energy is gone at

$$t_{1/2} = (0.693)(39\text{ s}) = 27\text{ s}$$

ASSESS The oscillator loses energy faster than it loses amplitude. This is what we should expect because the energy depends on the *square* of the amplitude.

STOP TO THINK 15.6 Rank in order, from largest to smallest, the time constants τ_a to τ_d of the decays shown in the figure. All the graphs have the same scale.



15.8 Driven Oscillations and Resonance

Thus far we have focused on the free oscillations of an isolated system. Some initial disturbance displaces the system from equilibrium, and it then oscillates freely until its energy is dissipated. These are very important situations, but they do not exhaust the possibilities. Another important situation is an oscillator that is subjected to a periodic external force. Its motion is called a **driven oscillation.**

A simple example of a driven oscillation is pushing a child on a swing, where your push is a periodic external force applied to the swing. A more complex example is a car driving over a series of equally spaced bumps. Each bump causes a periodic upward force on the car's shock absorbers, which are big, heavily damped springs. The electromagnetic coil on the back of a loudspeaker cone provides a periodic magnetic force to drive the cone back and forth, causing it to send out sound waves. Air turbulence moving across the wings of an aircraft can exert periodic forces on the wings and other aerodynamic surfaces, causing them to vibrate if they are not properly designed.

As these examples suggest, driven oscillations have many important applications. However, driven oscillations are a mathematically complex subject. We will simply hint at some of the results, saving the details for more advanced classes.

Consider an oscillating system that, when left to itself, oscillates at a frequency f_0. We will call this the **natural frequency** of the oscillator. The natural frequency for a mass on a spring is $\sqrt{k/m}/2\pi$, but it might be given by some other expression for another type of oscillator. Regardless of the expression, f_0 is simply the frequency of the system if it is displaced from equilibrium and released.

Suppose that this system is subjected to a *periodic* external force of frequency f_{ext}. This frequency, which is called the **driving frequency,** is completely independent of the oscillator's natural frequency f_0. Somebody or something in the environment selects the frequency f_{ext} of the external force, causing the force to push on the system f_{ext} times every second.

Although it is possible to solve Newton's second law with an external driving force, we will be content to look at a graphical representation of the solution. The most important result is that the oscillation amplitude depends very sensitively on the frequency f_{ext} of the driving force. The response to the driving frequency is shown in FIGURE 15.25 for a system with $m = 1.0$ kg, a natural frequency $f_0 = 2.0$ Hz, and a damping constant $b = 0.20$ kg/s. This graph of amplitude versus driving frequency, called the **response curve,** occurs in many different applications.

When the driving frequency is substantially different from the oscillator's natural frequency, at the right and left edges of Figure 15.25, the system oscillates but the amplitude is very small. The system simply does not respond well to a driving frequency that differs much from f_0. As the driving frequency gets closer and closer to the natural frequency, the amplitude of the oscillation rises dramatically. After all, f_0 is the frequency at which the system "wants" to oscillate, so it is quite happy to respond to a driving frequency near f_0. Hence the amplitude reaches a maximum when the driving frequency exactly matches the system's natural frequency: $f_{\text{ext}} = f_0$.

The amplitude can become exceedingly large when the frequencies match, especially if the damping constant is very small. FIGURE 15.26 shows the same oscillator with three different values of the damping constant. There's very little response if the damping constant is increased to 0.80 kg/s, but the amplitude for $f_{\text{ext}} = f_0$ becomes very large when the damping constant is reduced to 0.08 kg/s. This large-amplitude response to a driving force whose frequency matches the natural frequency of the system is a phenomenon called **resonance.** The condition for resonance is

$$f_{\text{ext}} = f_0 \quad \text{(resonance condition)} \tag{15.61}$$

Within the context of driven oscillations, the natural frequency f_0 is often called the **resonance frequency.**

An important feature of Figure 15.26 is how the amplitude and width of the resonance depend on the damping constant. A heavily damped system responds fairly

FIGURE 15.25 The response curve of a driven oscillator at frequencies near its natural frequency of 2.0 Hz.

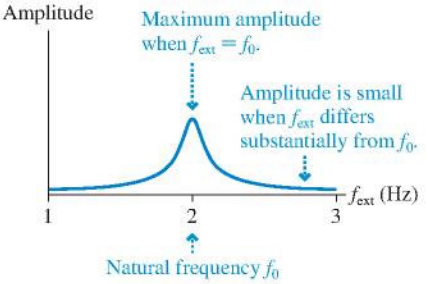
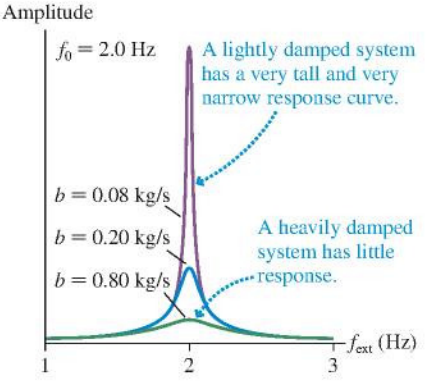


FIGURE 15.26 The resonance amplitude becomes higher and narrower as the damping constant decreases.



A singer or musical instrument can shatter a crystal goblet by matching the goblet's natural oscillation frequency.

little, even at resonance, but it responds to a wide range of driving frequencies. Very lightly damped systems can reach exceptionally high amplitudes, but notice that the range of frequencies to which the system responds becomes narrower and narrower as b decreases.

This allows us to understand why a few singers can break crystal goblets but not inexpensive, everyday glasses. An inexpensive glass gives a "thud" when tapped, but a fine crystal goblet "rings" for several seconds. In physics terms, the goblet has a much longer time constant than the glass. That, in turn, implies that the goblet is very lightly damped while the ordinary glass is heavily damped (because the internal forces within the glass are not those of a high-quality crystal structure).

The singer causes a sound wave to impinge on the goblet, exerting a small driving force at the frequency of the note she is singing. If the singer's frequency matches the natural frequency of the goblet—resonance! Only the lightly damped goblet, like the top curve in Figure 15.26, can reach amplitudes large enough to shatter. The restriction, though, is that its natural frequency has to be matched very precisely. The sound also has to be very loud.

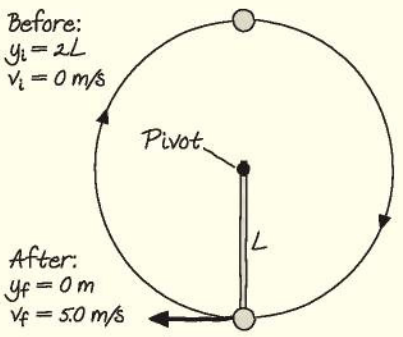
CHALLENGE EXAMPLE 15.11 A swinging pendulum

A pendulum consists of a massless, rigid rod with a mass at one end. The other end is pivoted on a frictionless pivot so that the rod can rotate in a complete circle. The pendulum is inverted, with the mass directly above the pivot point, then released. The speed of the mass as it passes through the lowest point is 5.0 m/s. If the pendulum later undergoes small-amplitude oscillations at the bottom of the arc, what will its frequency be?

MODEL This is a simple pendulum because the rod is massless. However, our analysis of a pendulum used the small-angle approximation. It applies only to the small-amplitude oscillations at the end, *not* to the pendulum swinging down from the inverted position. Fortunately, energy is conserved throughout, so we can analyze the big swing using conservation of mechanical energy.

VISUALIZE FIGURE 15.27 is a pictorial representation of the pendulum swinging down from the inverted position. The pendulum length is L, so the initial height is $2L$.

FIGURE 15.27 Before-and-after pictorial representation of the pendulum swinging down from an inverted position.



SOLVE The frequency of a simple pendulum is $f = \sqrt{g/L}/2\pi$. We're not given L, but we can find it by analyzing the pendulum's swing down from an inverted position. Mechanical energy is conserved, and the only potential energy is gravitational potential energy. Conservation of mechanical energy $K_f + U_{Gf} = K_i + U_{Gi}$, with $U_G = mgy$, is

$$\tfrac{1}{2}mv_f^2 + mgy_f = \tfrac{1}{2}mv_i^2 + mgy_i$$

The mass cancels, which is good since we don't know it, and two terms are zero. Thus

$$\tfrac{1}{2}v_f^2 = g(2L) = 2gL$$

Solving for L, we find

$$L = \frac{v_f^2}{4g} = \frac{(5.0\text{ m/s})^2}{4(9.80\text{ m/s}^2)} = 0.638\text{ m}$$

Now we can calculate the frequency:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} = \frac{1}{2\pi}\sqrt{\frac{9.80\text{ m/s}^2}{0.638\text{ m}}} = 0.62\text{ Hz}$$

ASSESS The frequency corresponds to a period of about 1.5 s, which seems reasonable.

SUMMARY

The goal of Chapter 15 has been to learn about systems that oscillate in simple harmonic motion.

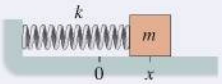
GENERAL PRINCIPLES

Dynamics

SHM occurs when a **linear restoring force** acts to return a system to an equilibrium position.

Horizontal spring

$(F_{net})_x = -kx$



Vertical spring

The origin is at the equilibrium position $\Delta L = mg/k$.

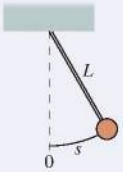
$(F_{net})_y = -ky$

Both: $\omega = \sqrt{\dfrac{k}{m}} \qquad T = 2\pi\sqrt{\dfrac{m}{k}}$

Simple pendulum

$\omega = \sqrt{\dfrac{g}{L}}$

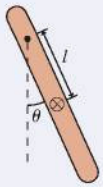
$T = 2\pi\sqrt{\dfrac{L}{g}}$



Physical pendulum

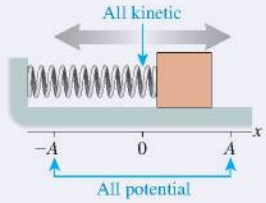
$\omega = \sqrt{\dfrac{Mgl}{I}}$

$T = 2\pi\sqrt{\dfrac{I}{Mgl}}$



Energy

If there is no friction or dissipation, kinetic and potential energy are alternately transformed into each other, but the total mechanical energy $E = K + U$ is conserved.

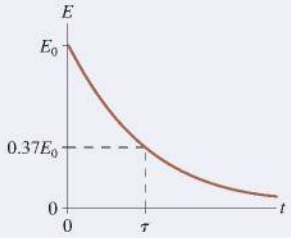


$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$
$$= \tfrac{1}{2}m(v_{max})^2$$
$$= \tfrac{1}{2}kA^2$$

The energy of a lightly damped oscillator decays exponentially

$$E = E_0 e^{-t/\tau}$$

where τ is the **time constant**.



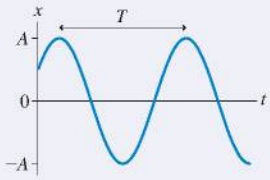
IMPORTANT CONCEPTS

Simple harmonic motion (SHM) is a sinusoidal oscillation with period T and amplitude A.

Frequency $f = \dfrac{1}{T}$

Angular frequency

$$\omega = 2\pi f = \frac{2\pi}{T}$$



Position $x(t) = A\cos(\omega t + \phi_0)$

$$= A\cos\left(\frac{2\pi t}{T} + \phi_0\right)$$

Velocity $v_x(t) = -v_{max}\sin(\omega t + \phi_0)$ with maximum speed $v_{max} = \omega A$

Acceleration $a_x(t) = -\omega^2 x(t) = -\omega^2 A\cos(\omega t + \phi_0)$

SHM is the projection onto the x-axis of **uniform circular motion.**

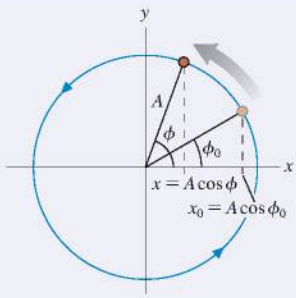
$\phi = \omega t + \phi_0$ is the **phase**

The position at time t is

$$x(t) = A\cos\phi$$
$$= A\cos(\omega t + \phi_0)$$

The **phase constant** ϕ_0 is determined by the initial conditions:

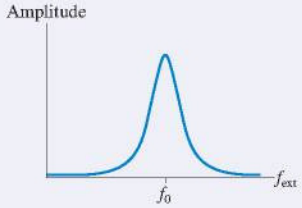
$$x_0 = A\cos\phi_0 \qquad v_{0x} = -\omega A\sin\phi_0$$



APPLICATIONS

Resonance

When a system is driven by a periodic external force, it responds with a large-amplitude oscillation if $f_{ext} \approx f_0$, where f_0 is the system's natural oscillation frequency, or **resonant frequency.**

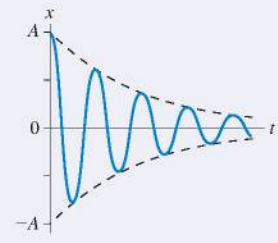


Damping

If there is a drag force $\vec{F}_{drag} = -b\vec{v}$, where b is the damping constant, then (for lightly damped systems)

$x(t) = Ae^{-bt/2m}\cos(\omega t + \phi_0)$

The time constant for energy loss is $\tau = m/b$.



TERMS AND NOTATION

oscillatory motion
period, T
simple harmonic motion, SHM
amplitude, A
frequency, f
hertz, Hz
angular frequency, ω
phase, ϕ
phase constant, ϕ_0
restoring force
equation of motion
small-angle approximation
linear restoring force
damped oscillation
damping constant, b
envelope
exponential decay
time constant, τ
half-life, $t_{1/2}$
driven oscillation
natural frequency, f_0
driving frequency, f_{ext}
response curve
resonance
resonance frequency, f_0

CONCEPTUAL QUESTIONS

1. A block oscillating on a spring has period $T = 2$ s. What is the period if:
   a. The block's mass is doubled? Explain. Note that you do not know the value of either m or k, so do *not* assume any particular values for them. The required analysis involves thinking about ratios.
   b. The value of the spring constant is quadrupled?
   c. The oscillation amplitude is doubled while m and k are unchanged?
2. A pendulum on Planet X, where the value of g is unknown, oscillates with a period $T = 2$ s. What is the period of this pendulum if:
   a. Its mass is doubled? Explain. Note that you do not know the value of m, L, or g, so do not assume any specific values. The required analysis involves thinking about ratios.
   b. Its length is doubled?
   c. Its oscillation amplitude is doubled?
3. FIGURE Q15.3 shows a position-versus-time graph for a particle in SHM. What are (a) the amplitude A, (b) the angular frequency ω, and (c) the phase constant ϕ_0?

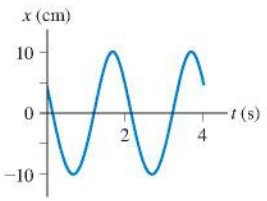


FIGURE Q15.3

4. FIGURE Q15.4 shows a position-versus-time graph for a particle in SHM.
   a. What is the phase constant ϕ_0? Explain.
   b. What is the phase of the particle at each of the three numbered points on the graph?

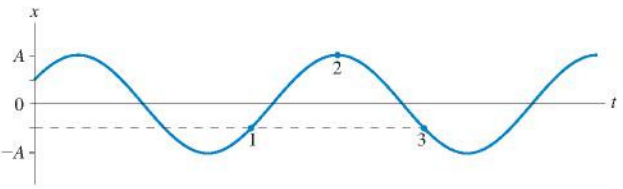


FIGURE Q15.4

5. Equation 15.25 states that $\frac{1}{2}kA^2 = \frac{1}{2}m(v_{max})^2$. What does this mean? Write a couple of sentences explaining how to interpret this equation.
6. A block oscillating on a spring has an amplitude of 20 cm. What will the amplitude be if the total energy is doubled? Explain.
7. A block oscillating on a spring has a maximum speed of 20 cm/s. What will the block's maximum speed be if the total energy is doubled? Explain.
8. The solid disk and circular hoop in FIGURE Q15.8 have the same radius and the same mass. Each can swing back and forth as a pendulum from a pivot at the top edge. Which, if either, has the larger period of oscillation?

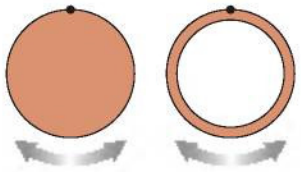


FIGURE Q15.8

9. FIGURE Q15.9 shows the potential-energy diagram and the total energy line of a particle oscillating on a spring.
   a. What is the spring's equilibrium length?
   b. Where are the turning points of the motion? Explain.
   c. What is the particle's maximum kinetic energy?
   d. What will be the turning points if the particle's total energy is doubled?

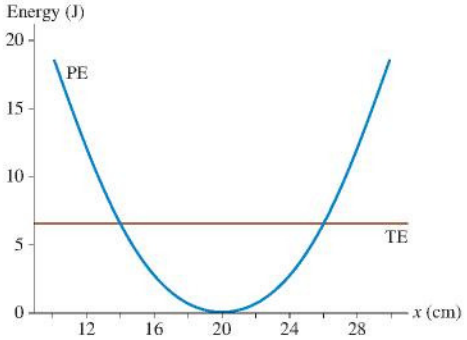


FIGURE Q15.9

10. Suppose the damping constant b of an oscillator increases.
    a. Is the medium more resistive or less resistive?
    b. Do the oscillations damp out more quickly or less quickly?
    c. Is the time constant τ increased or decreased?
11. a. Describe the difference between τ and T. Don't just *name* them; say what is different about the physical concepts they represent.
    b. Describe the difference between τ and $t_{1/2}$.
12. What is the difference between the driving frequency and the natural frequency of an oscillator?

EXERCISES AND PROBLEMS

Problems labeled integrate material from earlier chapters.

Exercises

Section 15.1 Simple Harmonic Motion

1. I An air-track glider attached to a spring oscillates between the 10 cm mark and the 60 cm mark on the track. The glider completes 10 oscillations in 33 s. What are the (a) period, (b) frequency, (c) angular frequency, (d) amplitude, and (e) maximum speed of the glider?
2. II An air-track glider is attached to a spring. The glider is pulled to the right and released from rest at $t = 0$ s. It then oscillates with a period of 2.0 s and a maximum speed of 40 cm/s.
   a. What is the amplitude of the oscillation?
   b. What is the glider's position at $t = 0.25$ s?
3. I When a guitar string plays the note "A," the string vibrates at 440 Hz. What is the period of the vibration?
4. II An object in SHM oscillates with a period of 4.0 s and an amplitude of 10 cm. How long does the object take to move from $x = 0.0$ cm to $x = 6.0$ cm?

Section 15.2 SHM and Circular Motion

5. I What are the (a) amplitude, (b) frequency, and (c) phase constant of the oscillation shown in FIGURE EX15.5?

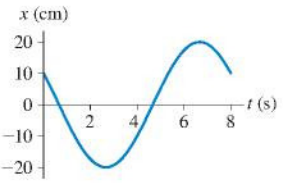


FIGURE EX15.5

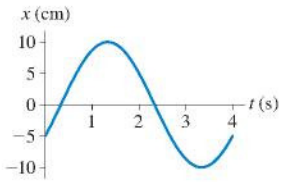


FIGURE EX15.6

6. II What are the (a) amplitude, (b) frequency, and (c) phase constant of the oscillation shown in FIGURE EX15.6?
7. II FIGURE EX15.7 is the position-versus-time graph of a particle in simple harmonic motion.
   a. What is the phase constant?
   b. What is the velocity at $t = 0$ s?
   c. What is v_{max}?

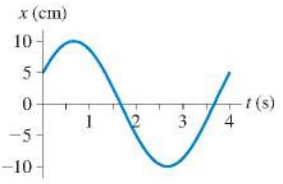


FIGURE EX15.7

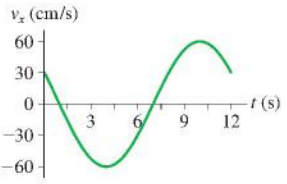


FIGURE EX15.8

8. II FIGURE EX15.8 is the velocity-versus-time graph of a particle in simple harmonic motion.
   a. What is the amplitude of the oscillation?
   b. What is the phase constant?
   c. What is the position at $t = 0$ s?
9. II An object in simple harmonic motion has an amplitude of 4.0 cm, a frequency of 2.0 Hz, and a phase constant of $2\pi/3$ rad. Draw a position graph showing two cycles of the motion.
10. II An object in simple harmonic motion has an amplitude of 8.0 cm, a frequency of 0.25 Hz, and a phase constant of $-\pi/2$ rad. Draw a position graph showing two cycles of the motion.
11. I An object in simple harmonic motion has amplitude 4.0 cm and frequency 4.0 Hz, and at $t = 0$ s it passes through the equilibrium point moving to the right. Write the function $x(t)$ that describes the object's position.
12. I An object in simple harmonic motion has amplitude 8.0 cm and frequency 0.50 Hz. At $t = 0$ s it has its most negative position. Write the function $x(t)$ that describes the object's position.
13. II An air-track glider attached to a spring oscillates with a period of 1.5 s. At $t = 0$ s the glider is 5.00 cm left of the equilibrium position and moving to the right at 36.3 cm/s.
   a. What is the phase constant?
   b. What is the phase at $t = 0$ s, 0.5 s, 1.0 s, and 1.5 s?

Section 15.3 Energy in SHM

Section 15.4 The Dynamics of SHM

14. I A block attached to a spring with unknown spring constant oscillates with a period of 2.0 s. What is the period if
   a. The mass is doubled?
   b. The mass is halved?
   c. The amplitude is doubled?
   d. The spring constant is doubled?
   Parts a to d are independent questions, each referring to the initial situation.
15. II A 200 g air-track glider is attached to a spring. The glider is pushed in 10 cm and released. A student with a stopwatch finds that 10 oscillations take 12.0 s. What is the spring constant?
16. II A 200 g mass attached to a horizontal spring oscillates at a frequency of 2.0 Hz. At $t = 0$ s, the mass is at $x = 5.0$ cm and has $v_x = -30$ cm/s. Determine:
   a. The period.
   b. The angular frequency.
   c. The amplitude.
   d. The phase constant.
   e. The maximum speed.
   f. The maximum acceleration.
   g. The total energy.
   h. The position at $t = 0.40$ s.
17. II The position of a 50 g oscillating mass is given by $x(t) = (2.0\text{ cm})\cos(10t - \pi/4)$, where t is in s. Determine:
   a. The amplitude.
   b. The period.
   c. The spring constant.
   d. The phase constant.
   e. The initial conditions.
   f. The maximum speed.
   g. The total energy.
   h. The velocity at $t = 0.40$ s.
18. II A 1.0 kg block is attached to a spring with spring constant 16 N/m. While the block is sitting at rest, a student hits it with a hammer and almost instantaneously gives it a speed of 40 cm/s. What are
   a. The amplitude of the subsequent oscillations?
   b. The block's speed at the point where $x = \frac{1}{2}A$?
19. II A student is bouncing on a trampoline. At her highest point, her feet are 55 cm above the trampoline. When she lands, the trampoline sags 15 cm before propelling her back up. For how long is she in contact with the trampoline?

20. || FIGURE EX15.20 is a kinetic-energy graph of a mass oscillating on a *very* long horizontal spring. What is the spring constant?

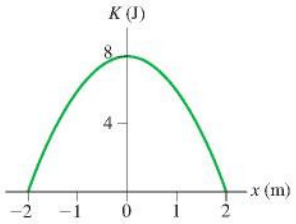


FIGURE EX15.20

Section 15.5 Vertical Oscillations

21. | A spring is hanging from the ceiling. Attaching a 500 g physics book to the spring causes it to stretch 20 cm in order to come to equilibrium.
    a. What is the spring constant?
    b. From equilibrium, the book is pulled down 10 cm and released. What is the period of oscillation?
    c. What is the book's maximum speed?
22. || A spring with spring constant 15 N/m hangs from the ceiling. A ball is attached to the spring and allowed to come to rest. It is then pulled down 6.0 cm and released. If the ball makes 30 oscillations in 20 s, what are its (a) mass and (b) maximum speed?
23. || A spring is hung from the ceiling. When a block is attached to its end, it stretches 2.0 cm before reaching its new equilibrium length. The block is then pulled down slightly and released. What is the frequency of oscillation?

Section 15.6 The Pendulum

24. | A grandfather clock ticks each time the pendulum passes through the lowest point. If the pendulum is modeled as a simple pendulum, how long must it be for the ticks to occur once a second?
25. || A 200 g ball is tied to a string. It is pulled to an angle of 8.0° and released to swing as a pendulum. A student with a stopwatch finds that 10 oscillations take 12 s. How long is the string?
26. | A mass on a string of unknown length oscillates as a pendulum with a period of 4.0 s. What is the period if
    a. The mass is doubled?
    b. The string length is doubled?
    c. The string length is halved?
    d. The amplitude is doubled?
    Parts a to d are independent questions, each referring to the initial situation.
27. | What is the length of a pendulum whose period on the moon matches the period of a 2.0-m-long pendulum on the earth?
28. | What is the period of a 1.0-m-long pendulum on (a) the earth and (b) Venus?
29. | Astronauts on the first trip to Mars take along a pendulum that has a period on earth of 1.50 s. The period on Mars turns out to be 2.45 s. What is the free-fall acceleration on Mars?
30. ||| A 100 g mass on a 1.0-m-long string is pulled 8.0° to one side and released. How long does it take for the pendulum to reach 4.0° on the opposite side?
31. || A uniform steel bar swings from a pivot at one end with a period of 1.2 s. How long is the bar?

Section 15.7 Damped Oscillations

Section 15.8 Driven Oscillations and Resonance

32. | A 2.0 g spider is dangling at the end of a silk thread. You can make the spider bounce up and down on the thread by tapping lightly on his feet with a pencil. You soon discover that you can give the spider the largest amplitude on his little bungee cord if you tap exactly once every second. What is the spring constant of the silk thread?
33. || The amplitude of an oscillator decreases to 36.8% of its initial value in 10.0 s. What is the value of the time constant?
34. || In a science museum, a 110 kg brass pendulum bob swings at the end of a 15.0-m-long wire. The pendulum is started at exactly 8:00 A.M. every morning by pulling it 1.5 m to the side and releasing it. Because of its compact shape and smooth surface, the pendulum's damping constant is only 0.010 kg/s. At exactly 12:00 noon, how many oscillations will the pendulum have completed and what is its amplitude?
35. | BIO Vision is blurred if the head is vibrated at 29 Hz because the vibrations are resonant with the natural frequency of the eyeball in its socket. If the mass of the eyeball is 7.5 g, a typical value, what is the effective spring constant of the musculature that holds the eyeball in the socket?
36. || A 350 g mass on a 45-cm-long string is released at an angle of 4.5° from vertical. It has a damping constant of 0.010 kg/s. After 25 s, (a) how many oscillations has it completed and (b) how much energy has been lost?

Problems

37. || The motion of a particle is given by $x(t) = (25\text{ cm})\cos(10t)$, where t is in s. What is the first time at which the kinetic energy is twice the potential energy?
38. || a. When the displacement of a mass on a spring is $\frac{1}{2}A$, what fraction of the energy is kinetic energy and what fraction is potential energy?
    b. At what displacement, as a fraction of A, is the energy half kinetic and half potential?
39. || For a particle in simple harmonic motion, show that $v_{max} = (\pi/2)v_{avg}$, where v_{avg} is the average speed during one cycle of the motion.
40. || A 100 g block attached to a spring with spring constant 2.5 N/m oscillates horizontally on a frictionless table. Its velocity is 20 cm/s when $x = -5.0$ cm.
    a. What is the amplitude of oscillation?
    b. What is the block's maximum acceleration?
    c. What is the block's position when the acceleration is maximum?
    d. What is the speed of the block when $x = 3.0$ cm?
41. || A 0.300 kg oscillator has a speed of 95.4 cm/s when its displacement is 3.00 cm and 71.4 cm/s when its displacement is 6.00 cm. What is the oscillator's maximum speed?
42. || BIO An ultrasonic transducer, of the type used in medical ultrasound imaging, is a very thin disk ($m = 0.10$ g) driven back and forth in SHM at 1.0 MHz by an electromagnetic coil.
    a. The maximum restoring force that can be applied to the disk without breaking it is 40,000 N. What is the maximum oscillation amplitude that won't rupture the disk?
    b. What is the disk's maximum speed at this amplitude?

43. || BIO Astronauts in space cannot weigh themselves by standing on a bathroom scale. Instead, they determine their mass by oscillating on a large spring. Suppose an astronaut attaches one end of a large spring to her belt and the other end to a hook on the wall of the space capsule. A fellow astronaut then pulls her away from the wall and releases her. The spring's length as a function of time is shown in FIGURE P15.43.
    a. What is her mass if the spring constant is 240 N/m?
    b. What is her speed when the spring's length is 1.2 m?

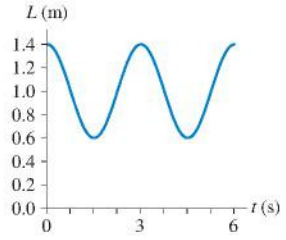


FIGURE P15.43

44. || Your lab instructor has asked you to measure a spring constant using a dynamic method—letting it oscillate—rather than a static method of stretching it. You and your lab partner suspend the spring from a hook, hang different masses on the lower end, and start them oscillating. One of you uses a meter stick to measure the amplitude, the other uses a stopwatch to time 10 oscillations. Your data are as follows:

Mass (g)	Amplitude (cm)	Time (s)
100	6.5	7.8
150	5.5	9.8
200	6.0	10.9
250	3.5	12.4

Use the best-fit line of an appropriate graph to determine the spring constant.

45. || A 5.0 kg block hangs from a spring with spring constant 2000 N/m. The block is pulled down 5.0 cm from the equilibrium position and given an initial velocity of 1.0 m/s back toward equilibrium. What are the (a) frequency, (b) amplitude, and (c) total mechanical energy of the motion?

46. ||| A 200 g block hangs from a spring with spring constant 10 N/m. At $t = 0$ s the block is 20 cm below the equilibrium point and moving upward with a speed of 100 cm/s. What are the block's
    a. Oscillation frequency?
    b. Distance from equilibrium when the speed is 50 cm/s?
    c. Distance from equilibrium at $t = 1.0$ s?

47. || A block hangs in equilibrium from a vertical spring. When a second identical block is added, the original block sags by 5.0 cm. What is the oscillation frequency of the two-block system?

48. || A 75 kg student jumps off a bridge with a 12-m-long bungee cord tied to his feet. The massless bungee cord has a spring constant of 430 N/m.
    a. How far below the bridge is the student's lowest point?
    b. How long does it take the student to reach his lowest point? You can assume that the bungee cord exerts no force until it begins to stretch.

49. || Scientists are measuring the properties of a newly discovered elastic material. They create a 1.5-m-long, 1.6-mm-diameter cord, attach an 850 g mass to the lower end, then pull the mass down 2.5 mm and release it. Their high-speed video camera records 36 oscillations in 2.0 s. What is Young's modulus of the material?

50. ||| A mass hanging from a spring oscillates with a period of 0.35 s. Suppose the mass and spring are swung in a horizontal circle, with the free end of the spring at the pivot. What rotation frequency, in rpm, will cause the spring's length to stretch by 15%?

51. || A compact car has a mass of 1200 kg. Assume that the car has one spring on each wheel, that the springs are identical, and that the mass is equally distributed over the four springs.
    a. What is the spring constant of each spring if the empty car bounces up and down 2.0 times each second?
    b. What will be the car's oscillation frequency while carrying four 70 kg passengers?

52. || The two blocks in FIGURE P15.52 oscillate on a frictionless surface with a period of 1.5 s. The upper block just begins to slip when the amplitude is increased to 40 cm. What is the coefficient of static friction between the two blocks?

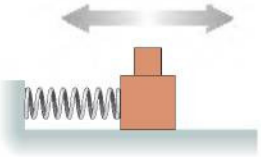


FIGURE P15.52

53. || A 1.00 kg block is attached to a horizontal spring with spring constant 2500 N/m. The block is at rest on a frictionless surface. A 10 g bullet is fired into the block, in the face opposite the spring, and sticks. What was the bullet's speed if the subsequent oscillations have an amplitude of 10.0 cm?

54. ||| BIO It has recently become possible to "weigh" DNA molecules by measuring the influence of their mass on a nano-oscillator. FIGURE P15.54 shows a thin rectangular cantilever etched out of silicon (density 2300 kg/m^3) with a small gold dot (not visible) at the end. If pulled down and released, the end of the cantilever vibrates with SHM, moving up and down like a diving board after a jump. When bathed with DNA molecules whose ends have been modified to bind with gold, one or more molecules may attach to the gold dot. The addition of their mass causes a very slight—but measurable—decrease in the oscillation frequency.

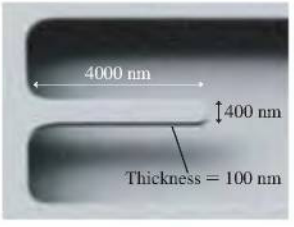


FIGURE P15.54

A vibrating cantilever of mass M can be modeled as a block of mass $\frac{1}{3}M$ attached to a spring. (The factor of $\frac{1}{3}$ arises from the moment of inertia of a bar pivoted at one end.) Neither the mass nor the spring constant can be determined very accurately— perhaps to only two significant figures—but the oscillation frequency can be measured with very high precision simply by counting the oscillations. In one experiment, the cantilever was initially vibrating at exactly 12 MHz. Attachment of a DNA molecule caused the frequency to decrease by 50 Hz. What was the mass of the DNA?

55. || It is said that Galileo discovered a basic principle of the pendulum—that the period is independent of the amplitude—by using his pulse to time the period of swinging lamps in the cathedral as they swayed in the breeze. Suppose that one oscillation of a swinging lamp takes 5.5 s.
    a. How long is the lamp chain?
    b. What maximum speed does the lamp have if its maximum angle from vertical is 3.0°?

56. || BIO Orangutans can move by *brachiation*, swinging like a pendulum beneath successive handholds. If an orangutan has arms that are 0.90 m long and repeatedly swings to a 20° angle, taking one swing after another, estimate its speed of forward motion in m/s. While this is somewhat beyond the range of validity of the small-angle approximation, the standard results for a pendulum are adequate for making an estimate.

57. || The pendulum shown in FIGURE P15.57 is pulled to a 10° angle on the left side and released.
   a. What is the period of this pendulum?
   b. What is the pendulum's maximum angle on the right side?

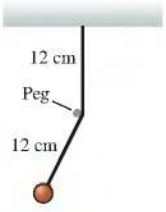


FIGURE P15.57

58. || A uniform rod of mass M and length L swings as a pendulum on a pivot at distance $L/4$ from one end of the rod. Find an expression for the frequency f of small-angle oscillations.

59. || BIO Interestingly, there have been several studies using cadavers to determine the moments of inertia of human body parts, information that is important in biomechanics. In one study, the center of mass of a 5.0 kg lower leg was found to be 18 cm from the knee. When the leg was allowed to pivot at the knee and swing freely as a pendulum, the oscillation frequency was 1.6 Hz. What was the moment of inertia of the lower leg about the knee joint?

60. || A 500 g air-track glider attached to a spring with spring constant 10 N/m is sitting at rest on a frictionless air track. A 250 g glider is pushed toward it from the far end of the track at a speed of 120 cm/s. It collides with and sticks to the 500 g glider. What are the amplitude and period of the subsequent oscillations?

61. || A 200 g block attached to a horizontal spring is oscillating with an amplitude of 2.0 cm and a frequency of 2.0 Hz. Just as it passes through the equilibrium point, moving to the right, a sharp blow directed to the left exerts a 20 N force for 1.0 ms. What are the new (a) frequency and (b) amplitude?

62. || FIGURE P15.62 is a top view of an object of mass m connected between two stretched rubber bands of length L. The object rests on a frictionless surface. At equilibrium, the tension in each rubber band is T. Find an expression for the frequency of oscillations *perpendicular* to the rubber bands. Assume the amplitude is sufficiently small that the magnitude of the tension in the rubber bands is essentially unchanged as the mass oscillates.

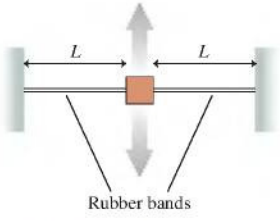


FIGURE P15.62

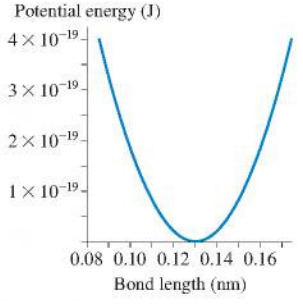


FIGURE P15.63

63. || A molecular bond can be modeled as a spring between two atoms that vibrate with simple harmonic motion. FIGURE P15.63 shows an SHM approximation for the potential energy of an HCl molecule. Because the chlorine atom is so much more massive than the hydrogen atom, it is reasonable to assume that the hydrogen atom ($m = 1.67 \times 10^{-27}$ kg) vibrates back and forth while the chlorine atom remains at rest. Use the graph to estimate the vibrational frequency of the HCl molecule.

64. || A penny rides on top of a piston as it undergoes vertical simple harmonic motion with an amplitude of 4.0 cm. If the frequency is low, the penny rides up and down without difficulty. If the frequency is steadily increased, there comes a point at which the penny leaves the surface.
   a. At what point in the cycle does the penny first lose contact with the piston?
   b. What is the maximum frequency for which the penny just barely remains in place for the full cycle?

65. || On your first trip to Planet X you happen to take along a 200 g mass, a 40-cm-long spring, a meter stick, and a stopwatch. You're curious about the free-fall acceleration on Planet X, where ordinary tasks seem easier than on earth, but you can't find this information in your Visitor's Guide. One night you suspend the spring from the ceiling in your room and hang the mass from it. You find that the mass stretches the spring by 31.2 cm. You then pull the mass down 10.0 cm and release it. With the stopwatch you find that 10 oscillations take 14.5 s. Based on this information, what is g?

66. ||| Suppose a large spherical object, such as a planet, with radius R and mass M has a narrow tunnel passing diametrically through it. A particle of mass m is inside the tunnel at a distance $x \leq R$ from the center. It can be shown that the net gravitational force on the particle is due entirely to the sphere of mass with radius $r \leq x$; there is no net gravitational force from the mass in the spherical shell with $r > x$.
   a. Find an expression for the gravitational force on the particle, assuming the object has uniform density. Your expression will be in terms of x, R, m, M, and any necessary constants.
   b. You should have found that the gravitational force is a linear restoring force. Consequently, in the absence of air resistance, objects in the tunnel will oscillate with SHM. Suppose an intrepid astronaut exploring a 150-km-diameter, 3.5×10^{18} kg asteroid discovers a tunnel through the center. If she jumps into the hole, how long will it take her to fall all the way through the asteroid and emerge on the other side?

67. || The 15 g head of a bobble-head doll oscillates in SHM at a frequency of 4.0 Hz.
   a. What is the spring constant of the spring on which the head is mounted?
   b. The amplitude of the head's oscillations decreases to 0.5 cm in 4.0 s. What is the head's damping constant?

68. || An oscillator with a mass of 500 g and a period of 0.50 s has an amplitude that decreases by 2.0% during each complete oscillation. If the initial amplitude is 10 cm, what will be the amplitude after 25 oscillations?

69. ||| A spring with spring constant 15.0 N/m hangs from the ceiling. A 500 g ball is attached to the spring and allowed to come to rest. It is then pulled down 6.0 cm and released. What is the time constant if the ball's amplitude has decreased to 3.0 cm after 30 oscillations?

70. ||| A captive James Bond is strapped to a table beneath a huge pendulum made of a 2.0-m-diameter uniform circular metal blade rigidly attached, at its top edge, to a 6.0-m-long, massless rod. The pendulum is set swinging with a 10° amplitude when its lower edge is 3.0 m above the prisoner, then the table slowly starts ascending at 1.0 mm/s. After 25 minutes, the pendulum's amplitude has decreased to 7.0°. Fortunately, the prisoner is freed with a mere 30 s to spare. What was the speed of the lower edge of the blade as it passed over him for the last time?

71. || A 250 g air-track glider is attached to a spring with spring constant 4.0 N/m. The damping constant due to air resistance is 0.015 kg/s. The glider is pulled out 20 cm from equilibrium and released. How many oscillations will it make during the time in which the amplitude decays to e^{-1} of its initial value?

72. || A 200 g oscillator in a vacuum chamber has a frequency of 2.0 Hz. When air is admitted, the oscillation decreases to 60% of its initial amplitude in 50 s. How many oscillations will have been completed when the amplitude is 30% of its initial value?
73. || Prove that the expression for $x(t)$ in Equation 15.55 is a solution to the equation of motion for a damped oscillator, Equation 15.54, if and only if the angular frequency ω is given by the expression in Equation 15.56.
74. || A block on a frictionless table is connected as shown in FIGURE P15.74 to two springs having spring constants k_1 and k_2. Show that the block's oscillation frequency is given by

$$f = \sqrt{f_1^2 + f_2^2}$$

where f_1 and f_2 are the frequencies at which it would oscillate if attached to spring 1 or spring 2 alone.

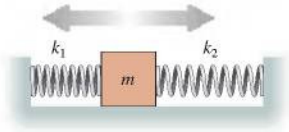


FIGURE P15.74

75. || A block on a frictionless table is connected as shown in FIGURE P15.75 to two springs having spring constants k_1 and k_2. Find an expression for the block's oscillation frequency f in terms of the frequencies f_1 and f_2 at which it would oscillate if attached to spring 1 or spring 2 alone.

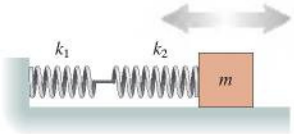


FIGURE P15.75

Challenge Problems

76. ||| A 15-cm-long, 200 g rod is pivoted at one end. A 20 g ball of clay is stuck on the other end. What is the period if the rod and clay swing as a pendulum?
77. ||| A solid sphere of mass M and radius R is suspended from a thin rod, as shown in FIGURE CP15.77. The sphere can swing back and forth at the bottom of the rod. Find an expression for the frequency f of small-angle oscillations.

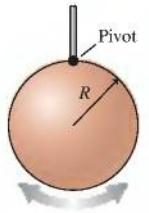


FIGURE CP15.77

78. ||| CALC A uniform rod of length L oscillates as a pendulum about a pivot that is a distance x from the center.
    a. For what value of x, in terms of L, is the oscillation period a minimum?
    b. What is the minimum oscillation period of a 15 kg, 1.0-m-long steel bar?
79. ||| A spring is standing upright on a table with its bottom end fastened to the table. A block is dropped from a height 3.0 cm above the top of the spring. The block sticks to the top end of the spring and then oscillates with an amplitude of 10 cm. What is the oscillation frequency?
80. ||| The analysis of a simple pendulum assumed that the mass was a particle, with no size. A realistic pendulum is a small, uniform sphere of mass M and radius R at the end of a massless string, with L being the distance from the pivot to the center of the sphere.
    a. Find an expression for the period of this pendulum.
    b. Suppose $M = 25$ g, $R = 1.0$ cm, and $L = 1.0$ m, typical values for a real pendulum. What is the ratio $T_{\text{real}}/T_{\text{simple}}$, where T_{real} is your expression from part a and T_{simple} is the expression derived in this chapter?
81. ||| FIGURE CP15.81 shows a 200 g uniform rod pivoted at one end. The other end is attached to a horizontal spring. The spring is neither stretched nor compressed when the rod hangs straight down. What is the rod's oscillation period? You can assume that the rod's angle from vertical is always small.

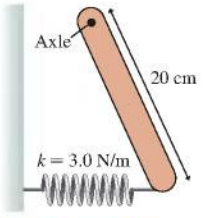


FIGURE CP15.81